Convergence of the Weiszfeld Algorithm for Weber Problems Using a Generalized "Distance" Function

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This paper considers a generalization of the single and multisource Weber problem for the case when the "distance" function is some power K of the usual l_p distance Properties of the generalized problem are established, and an appropriate generalization of the Weiszfeld iterative approach is given. A convergence proof is supplied for an ϵ-approximation to the original problem, under certain restrictions on p and K.

EARLY CONTRIBUTIONS to the facilities location literature favored the Euclidean metric for modeling distances. Later the rectangular metric came into vogue for approximating distances when movement is restricted to a network which is basically a rectangular grid. A selected bibliography of location analysis literature appears in Francis and Goldstein (1974). Both metrics are special cases of the l_p distance function

$$l_{pj}(x) = [\sum_{i=1}^{N} |x_i - a_{ji}|^p]^{1/p}, \qquad p \geq 1 \quad (1)$$

where x and a_j are points in N-dimensional space. This paper considers the generalization

$$f_{psj}(x) = [\sum_{i=1}^{N} |x_i - a_{ji}|^p]^{1/s}, \quad p \geq 1, \quad s > 0. \quad (2)$$

Love and Morris (1972, 1979) showed that (2) is quite accurate in estimating road travel distances from point references.

Alternatively, let $p/s \equiv K$ and $l_{pj}^K(x) \equiv [l_{pj}(x)]^K$, then $f_{psj}(x)$ is $l_{pj}^K(x)$. Cooper (1968) used l_{2j}^K to extend the generalized Weber problem. He suggested that in many applicatons $\phi = a \cdot d^K$ rather than $\phi = a \cdot d$ more accurately models cost (ϕ) in terms of distance (d), where a is the constant of proportionality. Figure 1, from Cooper (1968), illustrates economies and diseconomies that may be modeled. Cooper showed that the value of K can significantly affect the problem solution.

In the following, a generalization of the Weber problem is posed, an iterative solution algorithm is then developed, and a convergence proof

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is given. Finally, the multifacility problem is considered and convergence of an extended version of the algorithm is established.

1. FORMALIZATION

Formally, let $l_{pj}^K(x) \equiv [\sum_{t=1}^N |x_t - a_{jt}|^p]^{K/p}$, $p \geq 1$, $K > 0$. The following properties and Figure 1 would seem to imply that $l_p{}^K$ for $K > 1$ has less practical appeal than for $0 < K < 1$.

PROPERTY 1. *$l_p{}^K$ does not necessarily satisfy the triangle inequality when $K > 1$.*

PROPERTY 2. *$l_p{}^K$ satisfies the triangle inequality for $0 < K < 1$.*

Modeling distance and cost functions leads to an interest in solving the

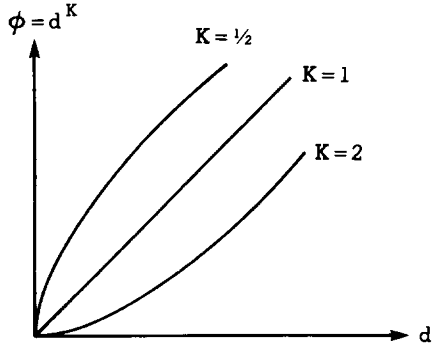

Figure 1. Cost functions as powers of the distance variable.

following generalization of the Weber problem

$$\text{minimize } Cl_p{}^K(x) = \sum_{j=1}^n w_j l_{pj}^K(x), \tag{3}$$

where x is the location of a "facility." The a_j, $j = 1, \cdots, n$ describe the location of the fixed "demand" points and $w_j > 0$ transforms $l_{pj}^K(x)$ into "cost." The literature of special cases of this problem is voluminous. Weiszfeld (1937) presented an iterative method for solving (3) when $p = 2$ and $K = 1$. Later related works are referenced in Morris and Verdini (1979) and Ostresh (1978). Ostresh proved global convergence for a modification of Weiszfeld's method. Most works concentrated on (3), with $p = 2$ and $K = 1$. Cooper (1968) applied the iterative approach when $p = 2$ and $K > 0$ and Katz (1969) studied the local convergence. Eyster et al. (1972) extended the approach to multifacility problems with $p = 1$ or 2 and $K = 1$, and Ostresh (1977) proved it to be a descent algorithm for $p = 2$. Morris and Verdini further extended the method to solving single and multifacility problems with $K = 1$ and $p \geq 1$. However, no complete proof of global convergence has previously been published for other than single facility problems with $p = 2$ and $K = 1$.

The following properties characterize problem (3).

Property 3. *$Cl_p{}^K(x)$ is convex when $K \geq 1$.*

Proof. $Cl_p{}^K(x)$ is a positive linear combination of $l_{pj}^K(x)$, $j = 1, \cdots, n$. When $K \geq 1$, $l_{pj}^K(x)$ is the composition of a monotone nondecreasing convex function with the convex l_p distance function.

Property 4. *The fixed points a_j, $j = 1, \cdots, n$ are local minima of $Cl_p{}^K(x)$ when $0 < K < 1$.*

Proof. Generalizing Cooper's (1968) proof for Euclidean distances, let $l_{pj}^K(x) = C_j(x)$. Choose a neighborhood about a_i such that for all $0 < \tau < \tau_i$, $\max_{j \neq i}(d/d\tau)C_j(a_i + \tau z) > -M_i$, where M_i is a positive real number and $\|z\|_p = 1$. Let $\|\cdot\|_p$ denote the l_p norm. Then $(d/d\tau)Cl_p{}^K(a_i + \tau z) = \sum_{j \neq i} w_j(d/d\tau)C_j(a_i + \tau z) + w_i(d/d\tau)C_i(a_i + \tau z) > -\sum_{j \neq i} w_j M_i + Kw_i/\tau^{(1-K)}$, since $C_i(a_i + \tau z) = \tau^K$. Choose $\underline{\tau} < \tau_i$ such that $Kw_i/\underline{\tau}^{(1-K)} > \sum_{j \neq i} w_j M_i$. Since $0 < K < 1$, $(d/d\tau)Cl_p{}^K(a_i + \tau z) > 0$ for $0 < \tau < \underline{\tau}$, which establishes the result.

The proof shows these minima become stronger as $K \to 0$. Since they occur at distinct points the following property holds.

Property 5. *$Cl_p{}^K(x)$ is neither convex nor concave when $0 < K < 1$.*

Figure 2 illustrates the local minima encountered when $0 < K < 1$. The example problem has fixed points in one-dimensional space at $a_j = j$, $j = 1, \cdots, 5$ and all $w_j = 1$ with $K = 0.25$, 0.50 and 0.75. As expected, the cusps become more pronounced as K is decreased.

2. THE ALGORITHM

To circumvent potential convergence difficulties caused by undefined quotients consider a differentiable approximation for $l_{pj}^K(x)$. Define $L_{pj}^K(x)$ as $[L_{pj}(x)]^K$, $p \geq 1$, $K > 0$, $\epsilon > 0$, where $L_{pj}(x) = \{\sum_{t=1}^N [(x_t - a_{jt})^2 + \epsilon]^{p/2}\}^{1/p}$ and ϵ is a smoothing constant. ϵ-approximation has been used in Eyster et al. (1972), Love (1969), Love and Morris (1975), Morris and Verdini (1979), and Wesolowsky and Love (1972).

Problem (3) is now approximated by

$$\text{minimize } CL_p{}^K(x) = \sum_{j=1}^n w_j L_{pj}^K(x). \tag{4}$$

Since $L_{pj}(x)$ is strictly convex (Morris and Verdini [1979]), so is $CL_p{}^K(x)$, for $K \geq 1$. Denote $\partial CL_p{}^K(x)/\partial x_t$ by $\nabla_t CL_p{}^K(x)$, the tth element of the gradient vector, then

$$\nabla_t CL_p{}^K(x) = \textstyle\sum_{j=1}^n w_j K(x_t - a_{jt})[(x_t - a_{jt})^2 + \epsilon]^{p/2-1}[L_{pj}(x)]^{K-p}.$$

Using $\nabla_t CL_p{}^K(x^*) = 0$, $t = 1, \cdots, N$ produces

$$x_t{}^* = (\textstyle\sum_j w_j[(x_t{}^* - a_{jt})^2 + \epsilon]^{p/2-1} \cdot [L_{pj}(x^*)]^{K-p} a_{jt}) / (\sum_j w_j[(x_t{}^* - a_{jt})^2 + \epsilon]^{p/2-1} \cdot [L_{pj}(x^*)]^{K-p}), \qquad t = 1, \cdots, N. \tag{5}$$

Replacing * on the left by $r + 1$ and on the right by r yields the iterative

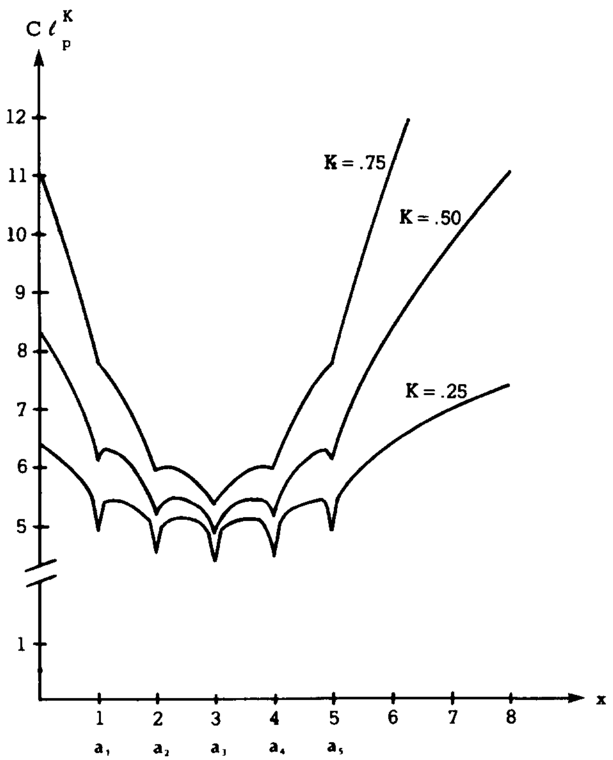

Figure 2. Examples of $Cl_p{}^K$ for $K < 1$.

scheme. See Morris and Verdini for a report on computational experience for $K = 1$. For the special case $p = 2$, $\epsilon = 0$ and $K = 1$, Katz (1974) has shown that, when x^* is a fixed point, the convergence rate can be linear, sublinear or quadratic; when x^* is not a fixed point convergence is always linear.

Using steps like those in Morris and Verdini gives

$$x_t^{r+1} = x_t{}^r - (K \textstyle\sum_j w_j[(x_t{}^r - a_{jt})^2 + \epsilon]^{p/2-1} \cdot [L_{pj}(x^r)]^{K-p})^{-1} \nabla_t CL_p{}^K(x^r), \tag{6}$$

or $$x^{r+1} = x^r - [M(x^r)]^{-1}\nabla CL_p{}^K(x^r).$$

$[M(x^r)]^{-1}$ is a diagonal matrix with positive diagonal elements and is therefore positive definite. This guarantees $-[M(x^r)]^{-1}\nabla CL_p{}^K(x^r)$ is a descent direction. The algorithmic map is

$$T{:}x \rightarrow T(x) = x - [M(x))]^{-1}\nabla CL_p{}^K(x), \tag{7}$$

with iterates given by $x^{r+1} = T(x^r)$. Termination might be warranted when either $|CL_p{}^K(x^r) - CL_p{}^K(T(x^r))|$ or $\|x^r - T(x^r)\|_p$ is small enough.

Wagner ([1975], see p. 551) suggests using a classical inequality to stop a gradient search technique. For $K \geq 1$, $CL_p{}^K(x)$ is convex and the inequality takes the form

$$CL_p{}^K(x^*) \geq CL_p{}^K(x^r) + \textstyle\sum_t \nabla_t CL_p{}^K(x^r)\cdot(x_t{}^* - x_t{}^r), \tag{8}$$

for any x^* and x^r, or

$$\textstyle\sum_t \nabla_t CL_p{}^K(x^r)\cdot(x_t^r - x_t{}^*) \geq CL_t{}^K(x^r) - CL_p{}^K(x^*).$$

Let $e_t > 0$ be such that $|x_t{}^r - x_t{}^*| < e_t$, and let x^* solve (4). Then

$$\textstyle\sum_t |\nabla_t CL_p{}^K(x^r)| e_t \geq CL_p{}^K(x^r) - CL_p{}^K(x^*) \geq 0, \tag{9}$$

which supplies a convenient bound on the suboptimality of the iterate x^r. The bound is convenient since $\nabla_t CL_p{}^K(x^r)$ is effectively calculated when computing x^{r+1}. By the proof of Lemma 4 below, e_t may be taken to be $\max_j |x_t{}^r - a_{jt}|$, $t = 1, \cdots, N$, which is also easily calculated as iterations are performed. The bound in (9) is similar to that suggested by Love and Yeong who pointed out the need for rational stopping rules in this context. Their bound (posed for $K = 1$) also accounts for the error induced by approximating the original problem. For $0 < K < 1$, $CL_p{}^K(x)$ is not convex. A rational stopping rule is lacking for this case.

The following property can be proved as was an analogous property in Morris and Verdini. It indicates a growing insensitivity to the original problem as ϵ is increased, but supports using the center of gravity as x^0.

Property 6. $\lim_{\epsilon\to\infty} x^* = \sum_j w_j a_j / \sum_j w_j$.

3. CONVERGENCE PROOF

Verdini (1976) has proven convergence for $p = 1$ or $p = 2$ and $K = 1$. The present proof is inspired by Kuhn (1973) whose work corrected Weiszfeld's original proof. One possible difficulty noted by Kuhn is that iterations may "overshoot." The following descent property guarantees that this will not happen.

Lemma 1. *If $T(x^r) \neq x^r$, then $CL_p{}^K(T(x^r)) < CL_p{}^K(x^r)$ for $1 \leq p \leq 2$, $0 < K \leq p$.*

Proof. Let $g(x) = \sum_j \omega_j[L_{pj}(x)]^p$, where $\omega_j = w_j[L_{pj}(x^r)]^{K-p}$. Then $g(x) = \sum_j \omega_j \sum_t[(x_t - a_{jt})^2 + \epsilon]^{p/2} = \sum_t \sum_j \omega_j[(x_t - a_{jt})^2 + \epsilon]^{p/2} = \sum_t g_t(x_t)$, say. Further, let $f_t(x_t) = \sum_j v_j[(x_t - a_{jt})^2 + \epsilon]$, where $v_j = \omega_j[(x_t^r - a_{jt})^2 + \epsilon]^{p/2-1}$. Since each $v_j > 0$, f_t is strictly convex. The unique minimum occurs at the center of gravity given by $\sum_j v_j a_{jt}/\sum_j v_j$, which is x_t^{r+1} by (5). So if $T(x^r) \neq x^r$, we have $f_t(x_t^{r+1}) \leq f_t(x_t^r) = g_t(x_t^r)$, and $f_t(x_t^{r+1}) < f_t(x_t^r)$ for at least one value of t.

On the other hand,

$$\begin{aligned} f_t(x_t^{r+1}) &= \sum_j \omega_j\{[(x_t^r - a_{jt})^2 + \epsilon]^{p/2}\}^{1-2/p}\{[(x_t^{r+1} - a_{jt})^2 + \epsilon]^{p/2}\}^{2/p} \\ &\geq \sum_j \omega_j\{(1 - 2/p)[(x_t^r - a_{jt})^2 + \epsilon]^{p/2} + (2/p)[(x_t^{r+1} - a_{jt})^2 + \epsilon]^{p/2}\} \\ &= (1 - 2/p)\, g_t(x_t^r) + (2/p)\, g_t(x_t^{r+1}). \end{aligned}$$

Equality holds for $p = 2$. The inequality holds for $1 \leq p < 2$ by the fundamental inequality $a^{1/p'}b^{1/q'} \geq a/p' + b/q'$ which holds for $1/p' + 1/q' = 1$, a, $b > 0$, $p' < 1$ and $p' \neq 0$. See Beckenbach and Bellman (1967), Equation 14.7. We have $a = [(x_t^{r+1} - a_{jt})^2 + \epsilon]^{p/2}$, $b = [(x_t^r - a_{jt})^2 + \epsilon]^{p/2}$, $p' = p/2$ and $q' = p/(p - 2)$. Putting these results together, $(1 - 2/p)\, g_t(x_t^r) + (2/p)\, g_t(x_t^{r+1}) \leq g_t(x_t^r)$, which means $g_t(x_t^{r+1}) \leq g_t(x_t^r)$ and the strict inequality holds for at least one value of t. This in turn means $g(T(x^r)) = \sum_t g_t(x_t^{r+1}) < \sum_t g_t(x_t^r) = g(x^r) = CL_p{}^K(x^r)$. Whereas,

$$\begin{aligned} g(T(x^r)) &= \sum_j w_j[L_{pj}(x^r)]^{K-p}[L_{pj}(T(x^r))]^p \\ &= \sum_j w_j[L_{pj}^K\,(x^r)]^{(K-p)/K}[L_{pj}^K\,(T(x^r))]^{p/K} \\ &\geq ((K-p)/K)CL_p{}^K(x^r) + (p/K)CL_p{}^K(T(x^r)), \quad \text{for} \quad K < p, \end{aligned}$$

$K \neq 0$, by the aforementioned fundamental inequality. Equality holds for $K = p$. Combining these results

$$((K-p)/K)CL_p{}^K(x^r) + (p/K)CL_p{}^K(T(x^r)) < CL_p{}^K(x^r)$$

and the assertion $CL_p{}^K(T(x^r)) < CL_p{}^K(x^r)$ is proven under the given conditions.

The second possible convergence difficulty noted by Kuhn is that the sequence x^r of generated points might remain at a nonoptimal fixed point. By design $CL_p{}^K(x)$ is differentiable and for $K \geq 1$ is strictly convex. Since $\nabla CL_p{}^K(x) \neq \vec{0}$ at a nonoptimal fixed point a_j, say, Equation (7) shows that $T(x^r) \neq x^r$ should $x^r = a_j$. For $0 < K < 1$, the fixed points are local minima of $Cl_p{}^K(x)$—which is approximated by $CL_p{}^K(x)$. Though convergence is still guaranteed, the convergence point is only guaranteed to be a stationary point of $CL_p{}^K(x)$.

LEMMA 2. *The algorithmic map defined by* (7) *is continuous.*

Proof. From (6) we may write (7) in the form $x_t^{r+1} = x_t^r - (1/m_t)$

$\nabla_t CL_p^K(x^r)$, $t = 1, \cdots, N$. The lemma holds since both $1/m_t$ and $\nabla_t CL_p^K(x)$ are continuous functions.

LEMMA 3. *$x^r = x^*$ if and only if $T(x^r) = x^r$, where x^* minimizes $CL_p^K(x)$ for $K \geq 1$ and x^* is a stationary point of $CL_p^K(x)$ for $0 < K < 1$.*

LEMMA 4. *$T(x^r)$ lies in a compact set.*

Proof. The algorithmic map can be expressed $x_t^{r+1} = \sum_J \alpha_{jt} a_{jt}$, $t = 1, \cdots, N$, with the obvious definition (see (5)) of α_{jt}. Clearly, all $\alpha_{jt} > 0$ and $\sum_J \alpha_{jt} = 1$, $t = 1, \cdots, N$ and the assertion is proved.

THEOREM 1. *Given any x^0, define x^r as $T^r(x^0)$ for $r = 1, 2, \cdots$. Then for $1 \leq p \leq 2$ and $0 < K \leq p$, $\lim_{r\to\infty} x^r = x^*$, where x^* minimizes $CL_p^K(x)$ for $K \geq 1$ and x^* is a stationary point of $CL_p^K(x)$ for $0 < K < 1$.*

Proof. By Lemma 4 we can invoke the Bolzano-Weierstrass Theorem which assures the existence of at least one point $\bar{x}$, say, and a convergent subsequence x^{r_l} such that $\lim_{l\to\infty} x^{r_l} = \bar{x}$. We must show that $\bar{x} = x^*$ in all cases.

If $T(x^r) = x^r$ for some r, the sequence repeats thereafter and $\bar{x} = x^r$. But then $\bar{x} = x^*$ by Lemma 3.

Otherwise, by Lemma 1, $CL_p^K(x^0) > CL_p^K(x^1) > \cdots > CL_p^K(x^r) > \cdots > CL_p^K(x^*)$. Hence

$$\lim_{r\to\infty}(CL_p^K(x^{r_l}) - CL_p^K(T(x^{r_l}))) = 0,$$

and since the algorithmic map is continuous by Lemma 2, we have

$$\lim_{l\to\infty} T(x^{r_l}) = T(\bar{x}), \text{ which implies } CL_p^K(\bar{x}) - CL_p^K(T(\bar{x})) = 0.$$

By Lemma 1, $\bar{x} = T(\bar{x})$ and $\bar{x} = x^*$ by Lemma 3 which establishes the theorem.

Approximating $Cl_p^K(x)$ by $CL_p^K(x)$ is an expedient to prevent quotients from being undefined. ϵ should only be large enough to serve this purpose, according to the computing accuracy and the scale of the problem data. This is because the minimizer of $CL_p^K(x)$ is not likely the minimizer of $Cl_p^K(x)$, as discussed by Verdini, and numerical evidence (see Eyster et al., Love and Morris (1975), and Morris and Verdini) for $K = 1$ indicates that as ϵ is reduced, the solution produced for the original problem is improved. If $T(x^r)$ is defined, the proof of Lemma 1 *remains valid* with $\epsilon = 0$. Therefore once iterations are terminated with $\epsilon > 0$, continued improvement in $Cl_p^K(x)$ is likely after setting $\epsilon = 0$ and *continuing* the iterations. A test for division by zero would then contribute to a stopping rule. The impediment to global convergence with $\epsilon = 0$, is that some quotient may be undefined prematurely. Indeed, Kuhn proved that for a denumerable number of x^0 Weiszfeld's original algorithm fails to converge to the minimizer of Cl_2^1. The quantitative question of how large or small to make ϵ is answered by the following property.

PROPERTY 7. *For* $1 \le p \le 2$ *and* $0 < K \le p$, $| CL_p{}^K(x) - Cl_p{}^K(x)| \le \delta(\epsilon) = N^{K/p}\epsilon^{K/2}\sum_{j=1}^{n} w_j$.

Proof. The inequality $(q_1 + q_2)^b \le q_1{}^b + q_2{}^b$ for $0 < b \le 1$ and q_1, $q_2 \ge 0$ is used twice. Using $q_1 = \sum_t [(x_t - a_{jt})^2 + \epsilon]^{p/2} - \sum_t [(x_t - a_{jt})^2]^{p/2} \ge 0$ and $q_2 = \sum_t [(x_t - a_{jt})^2]^{p/2} \ge 0$ with $0 < b = K/p \le 1$ produces $|L_{pj}^K(x) - l_{pj}^K(x)| \le |\sum_t [(x_t - a_{jt})^2 + \epsilon]^{p/2} - \sum_t [(x_t - a_{jt})^2]^{p/2}|^{K/p} = |D(x, \epsilon)|^{K/p}$, say. Second, $q_1 = (x_t - a_{jt})^2 \ge 0$ and $q_2 = \epsilon > 0$ with $\frac{1}{2} \le b = p/2 \le 1$ produces $D(x, \epsilon) \le N\epsilon^{p/2}$, which means $|L_{pj}^K(x) - l_{pj}^K(x)| \le N^{K/p}\epsilon^{K/2}$, and the conclusion follows directly.

Property 7 implies $CL_p{}^K(x)$ is uniformly convergent to $Cl_p{}^K(x)$ as $\epsilon \to 0$. Furthermore, assume $\hat{x}$ and x^* are global minimizers of $Cl_p{}^K(x)$ and $CL_p{}^K(x)$ respectively, but problem (4) is solved and $x^*$is produced. Then $Cl_p{}^K(x^*) - Cl_p{}^K(\hat{x}) \le CL_p{}^K(x^*) - Cl_p{}^K(\hat{x}) \le CL_p{}^K(\hat{x}) - Cl_p{}^K(\hat{x}) \le \delta(\epsilon)$, using the construction in Love and Yeong, and the suboptimality of x^* in problem (3) is bounded á priori.

Cooper (1968) performed computational experiments for the special case $p = 2$, $\epsilon = 0$, $N = 2$ and $0 < K < 1$. The algorithm frequently converged to the global optimum, even when diverse starting points were used. Viewing Figure 2 and the local minima for decreasing K values, it may be hypothesized that the algorithm will more likely become entrapped at a nonoptimal local minimum as $K \to 0$. The following numerical results support this hypothesis.

Let $n = 3$ with $a_1 = (0, 0)$, $a_2 = (1, 0)$, $a_3 = (0, 1)$, $w_1 = 1$, $w_2 = 10$, $w_3 = 1$, $p = 2$ and $K = 0.5$. Using $x^0 = (0.001, 0.001)$ the algorithm converged to $x^7 = (0, 0)$, a local minimum. However, with $K = 0.9$ and the same x^0, the global minimizer was obtained as $x^{12} = (1, 0)$. With $K = 0.5$ using the center of gravity as x^0 the convergence point was $x^6 = (1, 0)$, emphasizing the need to try different starting points when $K < 1$. Results were obtained with $\epsilon = 0$.

Results in Table I are for the eight configurations of fixed points given by Cooper (1963), where $N = 2$, $n = 7$, using $p = 1.5$, $w_j = j$ and three different values of K. Column triples are $Cl_{1.5}^K(\hat{x})$, $(\hat{x}_1, \hat{x}_2)$ followed by $Cl_{1.5}^K(T^r(x^z))$, $T^r(x^z)$, r and then $Cl_{1.5}^K(T^r(x^c))$, $T^r(x^c)$, r where $x^z = (0, 0)$ and x^c is the center of gravity. $\hat{x}$ is the global minimizer found by searching a rectangular grid (which included the fixed points) of gridwidth 0.1. All results were obtained with $\epsilon = 0$. When the algorithm terminated due to division by zero the convergence point was verified by repeating with a small positive value for ϵ. (In each such case a somewhat inferior solution was produced with $\epsilon > 0$.) Iterations were terminated when $Cl_{1.5}^K(x^{r-1}) - Cl_{1.5}^K(x^r) < 10^{-6}$. The unlikely starting point of x^z

frequently led to the global minimizer, as did x^c. Suboptimal results for $K = 0.15$ were the rule.

4. MULTIFACILITY LOCATION

The extension to locating m interrelated facilities, with L_p approximating l_p, is given by

$$\text{minimize } GL_p{}^K(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} L_{pj}^K(x_i) + \sum_{i=1}^{m-1} \sum_{g=i+1}^{m} v_{ig} L_p{}^K(x_i, x_g) \tag{10}$$

TABLE I

NUMERICAL RESULTS FOR $0 < K < 1$

Case	$K = 0.15$	$K = 0.50$	$K = 0.85$
1	31.91, (22, 29)	85.41, (22, 29)	217.28, (22.3, 22.7)
	39.94, (15, 15), 16*	91.89, (22.62, 23.26), 58*	217.28, (22.29, 22.68), 49
	31.91, (22, 29), 19	91.89, (22.62, 23.26), 29*	217.28, (22.29, 22.68), 39
2	33.75, (36, 32)	102.79, (20, 20)	287.72, (20, 20)
	37.05, (20, 20), 8*	102.79, (20, 20), 11	287.87, (21.26, 20.53), 72*
	37.05, (20, 20), 8*	102.79, (20, 20), 13	287.87, (21.27, 20.54), 122*
3	31.62, (35, 31)	84.00, (35, 26)	238.07, (35, 26)
	39.54, (5, 26), 16*	84.00, (35, 26), 14	238.07, (35, 26), 28
	32.26, (35, 26), 7*	84.00, (35, 26), 10	238.07, (35, 26), 24
4	32.90, (16, 12)	85.98, (26, 23)	213.99, (21.1, 23.8)
	36.78, (15, 23), 12*	85.98, (26, 23), 37	213.99, (21.07, 23.83), 26
	34.13, (26, 23), 12*	85.98, (26, 23), 27	213.99, (21.08, 23.83), 29
5	31.11, (24, 17)	70.55, (24, 17)	162.63, (24, 17)
	32.65, (17, 22), 27*	77.66, (17.88, 19.05), 23*	163.94, (20.90, 18.42), 186*
	31.11, (24, 17), 11	70.55, (24, 17), 18	163.94, (20.84, 18.44), 112*
6	32.53, (27, 17)	80.73, (20, 17)	200.62, (20, 17)
	33.33, (20, 17), 11*	80.73, (20, 17), 13	200.62, (20, 17), 26
	33.33, (20, 17), 6*	80.73, (20, 17), 9	200.62, (20, 17), 23
7	32.60, (11, 33)	79.82, (18, 23)	195.55, (18, 23)
	33.26, (18, 23), 6*	79.82, (18, 23), 7	195.55, (18, 23), 12
	33.28, (18, 23), 5*	79.82, (18, 23), 7	195.55, (18, 23), 13
8	35.64, (39, 33)	103.23, (33, 23)	291.79, (33, 17)
	35.85, (6, 10), 9*	104.05, (33, 17), 13*	291.79, (33, 17), 23
	38.13, (33, 17), 8*	104.05, (33, 17), 12*	291.79, (33, 17), 24

* Denotes a suboptimal point was found.

where $L_p{}^K(x_i, x_g) = \{\sum_{t=1}^{N} [(x_{it} - x_{gt})^2 + \epsilon]^{p/2}\}^{K/p}$ and the constants w_{ij} and v_{ig} are nonnegative. The argument x of $GL_p{}^K$ is now an $m \cdot N$-vector. For a well stated problem we will assume the facilities are chained, as defined by Francis and Cabot (1972). Facilities i and g are said to have an exchange whenever $v_{ig} > 0$ or $v_{gi} > 0$. Facility i is said to be chained if there exists a sequence of distinct facilities $i, i_1, \cdots, i_q$, such that there is an exchange between i and i_1, i_1 and i_2, $\cdots$, i_{q-1} and i_q, and there is some value of $j, j = 1, \cdots, n$ for which $w_{i_q j} > 0$.

$GL_p{}^K(x)$ is convex when $K \geq 1$. The counterpart of (5) is[1]

$$x_{gt}^{r+1} = \left(\textstyle\sum_j A_{gjt}(x_g{}^r)a_{jt} + \sum_{i\neq g} B_{git}(x_g{}^r, x_i^{r+\delta_i})x_{it}^{r+\delta_i}\right) / \left(\textstyle\sum_j A_{gjt}(x_g{}^r) + \sum_{i\neq g} B_{git}(x_g{}^r, x_i^{r+\delta_i})\right), \qquad t = 1, \cdots, N;\ g = 1, \cdots, m, \tag{11}$$

where

$$A_{gjt}(x_g{}^r) = w_{gj}[L_{pj}(x_g{}^r)]^{K-p}[(x_{gt}^r - a_{jt})^2 + \epsilon]^{p/2-1},$$

$$B_{git}(x_g{}^r, x_i^{r+\delta_i}) = v_{gi}[L_p(x_g{}^r, x_i^{r+\delta_i})]^{K-p}[(x_{gt}^r - a_{jt})^2 + \epsilon]^{p/2-1},$$

$$\textstyle\sum_{i\neq g} \text{ denotes } \sum_{i=1,i\neq g}^m \quad \text{and} \quad \delta_i = \begin{cases} 1, & i < g \\ 0, & i \geq g. \end{cases}$$

The algorithmic map is the following. For a given value of g, (11) is applied for $t = 1, \cdots, N$. Then x_g^{r+1} replaces $x_g{}^r$, i.e., δ_g changes from 0 to 1. This is done in order for $g = 1, \cdots, m$. When x_m^{r+1} is determined the process begins anew, unless a termination condition is satisfied. Ostresh (1977) also suggested this realization of the multifacility algorithm and proved a descent algorithm in studying the Euclidean distance, $K = 1$ case. Using a process similar to that used to derive (6) yields

$$x_{gt}^{r+1} = x_{gt}^r - \left(K\left[\textstyle\sum_j A_{gjt}(x_g{}^r) + \sum_{i\neq g} B_{git}(x_g{}^r, x_i^{r+\delta_i})\right]\right)^{-1} \nabla_{gt} GL_p{}^K(x_1^{r+\delta_1}, \cdots, x_m^{r+\delta_m})$$

and

$$x_g^{r+1} = x_g{}^r - [M(x_1^{r+\delta_1}, \cdots, x_m^{r+\delta_m})]^{-1} \nabla_g GL_p{}^K(x_1^{r+\delta_1}, \cdots, x_m^{r+\delta_m}).$$

The algorithm can be characterized as the repeated application of

$$T_g : x_g \rightarrow T_g(x_g) = x_g - [M(x_1, \cdots, x_m)]^{-1} \nabla_g GL_p{}^K(x_1, \cdots, x_m).$$

Now convergence of the algorithm in the single facility case implies convergence of the algorithm suggested here for the multifacility case. This is because applying T_g is equivalent to applying T since all facilities except the gth may temporarily be considered fixed points. This means T_g shares the descent property of T (defined by Lemma 1). The gth facility location is updated accordingly, then g is incremented by one and the process is repeated; but the descent property holds at each step. Since $GL_p{}^K$ is reduced at each step a lemma analogous to Lemma 1 could be stated for each cycle of updating all facility locations. Then arguments entirely analogous to those used to prove convergence of the algorithm in the single facility case can be drawn up to imply sufficiency of the descent property in the multifacility case.

As done by Morris and Verdini for $K = 1$ an analog of Property 6 can

[1] Whenever $r > i$, w_{2ri} is to be evaluated as w_{2ir}.

be developed as a starting point for minimizing $GL_p{}^K$. The analog of Property 7 has

$$\delta(\epsilon) = N^{K/p}\epsilon^{K/2}\left(\textstyle\sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} + \sum_{i=1}^{m-1} \sum_{g=i+1}^{m} v_{ij}\right).$$

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