Which Electoral Formula Is the Most Proportional? A New Look with New Evidence

Kenneth Benoit
Department of Political Science, Trinity College, Dublin 2, Ireland
e-mail: kbenoit@tcd.ie, http://benoit.tcd.ie

A ranking exists in electoral systems research of different electoral formulas—the mathematical functions governing the conversion of votes into legislative seats—in terms of both proportionality of seats and votes and favorability to the largest party. I reexamine this issue with new methods and new evidence, attempting to cross-validate previous rankings using a larger and more controlled data set and more precise parametric methods than have been applied previously. The results by and large confirm previous knowledge but also illuminate several important new facets obscured in previous investigations. For example, at common ranges of district magnitude (from 5 to 15 seats), it is shown that electoral formula may matter at least as much as district magnitude in shaping proportionality.

1 Introduction

ELECTORAL FORMULAS—defined as the mathematical mechanisms governing the transformation of votes into seats—have profound political consequences. A significant and ongoing issue in research on electoral rules concerns *proportionality*: To what degree do different electoral rules, especially variants of proportional representation formulas, affect the correspondence of vote shares with shares of legislative seats? Of importance to scholars and practical institution designers alike, the answer to this question may determine which variant of proportional representation (PR) formula is warranted for a given electoral system, or help to explain legislative outcomes in different contexts. Ranking PR formulas has been approached both theoretically (Gallagher 1992; Lijphart 1986; Loosemore and Hanby 1971) and empirically (Gallagher 1991; Blondel 1969), yet agreement is not universal. The most widely accepted ranking is Lijphart's (1986), which considers the Hare and Droop largest remainder (LR) methods to be the most proportional, followed by the Sainte-Laguë highest-average (HA) method, followed by Imperiali LR, d'Hondt HA, and Imperiali HA.[1]

In this paper I critically reexamine the ranking of electoral formulas by testing a wide variety of electoral formulas with regard to relative proportionality and, secondarily, with

Author's note: Thanks go to Gábor Tóka and Michael Gallagher for comments on early drafts of this paper and to Péter Kovács at the OZSH, Budapest, for his invaluable assistance in compiling these local election data. A replication data set for the analysis contained in this article is available at the Political Analysis website.

[1] Lijphart also included the single-transferable vote method which I do not consider here since, unlike all of the other methods, it involves an ordinal ballot structure. Technical details and terminology of electoral formulas are not explained here; for a detailed treatment see Gallagher (1992).

regard to their advantageousness for the largest party. The study contributes to the understanding of electoral formulas in several ways. First, it draws on a data set of examples much larger than found in previous research: a total of 4274 election results varying in both district magnitude and electoral formula. Each observed election comes from a single district, each taking place in the same country on the same day to avoid possible heterogeneity problems from cross-national pooling. Second, the approach to analyzing this data explicitly parameterizes proportionality and uses regression analysis to produce more precise estimates than any previous study of this topic. The results therefore permit us to refine existing rankings as well as uncover a few unexpected differences with regard to the Sainte-Laguë formula. Third, the findings underscore an important distinction between the proportionality of a system and its favorability to large parties. Finally, controlling for the partial effects of formula versus district magnitude (M) demonstrates that for many meaningful ranges of district magnitudes, the relative effects of formula are at least as significant as district magnitude. This finding adds an important qualification to previous work that tends to emphasize district magnitude as the most important determinant of proportionality (Taagepera and Shugart 1989; Gallagher 1991, p. 50; Taagepera and Laasko 1980, p. 443).

2 Data and Methods

The data set for comparing electoral formulas consists of simulated election results generated from actual vote distributions. The votes are divided into two classes: those taking place originally in single-member districts (SMDs) and those occurring in multimember districts where a form of PR was used. The votes come from elections to Hungarian local bodies held in December 1994 (see Benoit 2000). Only a subset of the 10,387 elected bodies in that data set is used here, namely, the 2063 single-member district elections as well as the 162 municipal councils and 39 county assemblies elected through proportional representation. The district magnitudes of the municipal assemblies range from 3 to 15, while the county assembly district sizes range from 5 to 66 (see Table 1). For the 201 PR elections, $201 \cdot 11 = 2211$ simulated PR election outcomes were generated from the actual votes. The simulated outcomes employ 11 election formulas: the Adams, equal proportions, Danish, Imperiali, d'Hondt, and Sainte-Laguë and modified Sainte-Laguë HA methods, as well as the Hare, Droop, and Imperiali largest-remainder (LR) methods (see Gallagher 1991, 1992). The Hungarian Sainte-Laguë formula (the classic Sainte-Laguë series with an initial divisor of 1.5) is also included since it was the rule actually applied to these votes in Hungary. This

Table 1 District magnitude frequency among the sample

District magnitude	*Frequency*
1	2063
2–5	1
6–9	126
10–15	41
16–20	12
21–30	12
31–40	7
41–66	2
Total	2264

selection provides a final sample of 4274 elections for use in comparing proportionality profiles: 2211 PR elections for comparing proportionality and 2063 SMD-plurality elections to act as a baseline.

The choice of Hungarian local elections for the sample of vote distributions is especially suited to the comparison of electoral formulas. First, district magnitudes in the data set vary considerably over a meaningful range of values. The districts summarized in Table 1 have an average magnitude of 11.5, with a standard deviation of 8.16. These ranges are highly comparable to district magnitudes in other national contexts. For example, 66 of the 72 democracies surveyed by Cox (1997, p. 55) had median primary electoral district magnitudes of 15 or less. Likewise, median district magnitudes from Amorim-Neto and Cox's (1997) data set of 54 electoral systems had a median value of 12.32. Clusters of similar values also fill the PR examples in Lijphart's (1994) survey of 27 democracies. Second, the distribution of votes also varies considerably. Since these votes were taken from actual elections, furthermore, they are likely to be set at realistic values. For example, the average effective number of parties was 5.31 in the PR elections and 4.17 in the single-member district elections. Overall the districts averaged 4.76 effective parties, with a standard deviation of 1.41, meaning that 95% of the elections examined contained between 3.35 and 6.17 effective parties. The main advantage of this approach to votes is that formula and district magnitude vary while the distribution of votes is held constant, since it is the same for each electoral formula. How these votes are determined is irrelevant for the purposes of the mechanical effects of proportionality; the key is to have variation in the distribution of votes. This is because proportionality and favorability to large parties are mechanical features whose *mechanical effect* is separate from the issue of the distribution of votes, concerning instead only the transformation of votes into seats.

Two summary characterizations of disproportionality are used in this analysis, used to measure each PR formula's disproportionality and advantageousness to the largest party, respectively.

DISPRLS: Gallagher's least-squares disproportionality index, ranging from 0 to 100, similar to the well-known Loosemore–Hanby (1971) index but registering small discrepancies less than large ones (Gallagher 1991). It is calculated as $\sqrt{\frac{1}{2}\sum_i (v_i - s_i)^2}$, and ranges from 0 to 100. A zero indicates perfect proportionality, and a 100 means that a candidate with no votes won a seat.

BONUSRAT: The bonus ratio of seats to votes awarded to the party winning the largest number of votes, calculated as the percentage of seats won by the largest party divided by the percentage of votes cast for the largest party. This measure is identical to the "advantage ratio" of Taagepera and Laasko (1980) applied to the largest party.

Unlike most previous empirical comparisons of the differences between PR formulas, I use a regression model to compare differences in proportionality, interpreting each formula chiefly on the basis of its partial coefficients. The results that follow are based on two regressions, one for each of the two disproportionality indexes. The model includes an intercept that represents the baseline case of the single-member district elections, where $M = 1$, and dummy variables both plain and interactive with M for each PR formula. The model employs the curvilinear function used often in previous studies of the effect of district magnitude on the number of parties. The logarithmic model assumes that the marginal effect of district magnitude on proportionality decreases as the district magnitude increases. To

Table 2 Comparing the proportionality of different electoral formulas[a]

	Combined regression coefficients			
Formula	*Constant*	log M	*Lijphart (1986) ranking*	
St.-Laguë HA	13.13	−6.526	Hare LR	*Most proportional*
Hare LR	13.43	−6.954	Droop LR	↑
Droop LR	13.68	−7.023	St.-Laguë HA	
Danish HA	14.28	−7.355		
Imperiali LR	14.86	−7.680	Imperiali LR	
Modified St.-Laguë HA	15.07	−7.828		
Hungarian St.-Laguë HA	15.99	−8.502		
d'Hondt HA	16.61	−8.293	d'Hondt HA	
Imperiali HA	24.16	−10.722	Imperiali HA	
Equal proportions HA	35.94	−21.029		
Adams HA	36.04	−20.961		↓
Plurality	50.44	—	Plurality	*Least proportional*

[a]Dependent variable: DISPRLS. $n = 4274, \sigma = 7.47$, $R^2 = .89$. All coefficients significant at the $p < .01$ level.

accomplish this I use the (base 10) logarithm log M instead of the simple value of district magnitude in all estimations, consistent with previous research (e.g., Amorim-Neto and Cox 1997; Taagepera and Shugart 1989).

3 Comparing Proportionality

Table 2 presents the results of a regression of the pooled sample of 4274 elections [$11 \cdot (201$ PR$) + 2063$ city SMDs] on log M. The regression model contains 23 parameters (in addition to σ): an intercept representing the baseline case of SMD elections, where $M = 1$, plus both plain and interactive β parameters for each of the 11 PR types. For ease of interpretation, the results in Table 2 present the combined intercept and slope estimates[2] for each type of PR as well as the summary statistics for each regression. Each formula is therefore associated with two parameters: an intercept and a slope. The intercept parameter may be interpreted as the disproportionality of the system as it approaches a district size of 1, yielding an equivalence with plurality. It represents the starting point or baseline average disproportionality of the formula. The slope parameter measures the responsiveness in terms of proportionality of the formula under conditions of increasing district magnitude. In the context of this model, larger negative slopes indicate a more rapid convergence to proportionality as the district magnitude approaches infinity.

The findings presented in Table 2 are largely consistent with Lijphart's ranking, with the exception of Sainte-Laguë discussed below. The results reaffirm that the Imperiali HA and d'Hondt are among the least proportional formulas (aside from the rarely used Adams and equal Proportions HA methods), relative to the largest remainder methods and

[2]"Combined" in this context means simply that the coefficients for the dummy variables have been added to the baseline coefficients to make interpretation easier. Full "raw" regression results are available on-line in the replication data set.

to Sainte-Laguë (Lijphart 1986; Loosemore and Hanby 1971). The findings also highlight the distinction between two sources of disproportionality: awarding the largest parties more than their proportional share of seats and awarding the smallest parties more than their proportional share of seats. The Adams and equal proportions HA formulas—ranked the least proportional in Table 2—both guarantee even the smallest of parties one seat each. (The Adams method, for example, is used for apportioning seats to the U.S. House of Representatives to states on the basis of population to guarantee that each state has at least one representative.) The result is a high score on the disproportionality index because parties with small or possibly even no votes receive seats significantly greater than their vote share. This source of disproportionality is quite different, however, from that driving the Imperiali HA result, which favors large parties. There is a difference, therefore, between the disproportionality of an electoral rule and its favorability toward large parties, since disproportionality may stem from the overrepresentation either of small parties or of large ones. This difference is explored more in the next section.

The estimates of the parameters of slope and intercept also confirm the utility of this approach to characterizing proportionality. The ranking of estimates of $\log M$ for each formula is, with only one exception, a perfect inverse ranking of the estimates of the constant for each formula. Figure 1 portrays the eventual convergence of every formula toward perfect proportionality at higher levels of district magnitude. After the Adams and equal proportions HA formulas, the other formulas reach perfect proportionality between $M = 75$ and $M = 88$, and the d'Hondt and classic Sainte-Laguë formulas at around $M = 105$. After the Adams and equal proportions HA formulas, the Hungarian Sainte-Laguë is the first to reach perfect proportionality, at around 76 seats. These findings indicate that while all PR methods eventually converge to proportionality, the rate at which they do so as

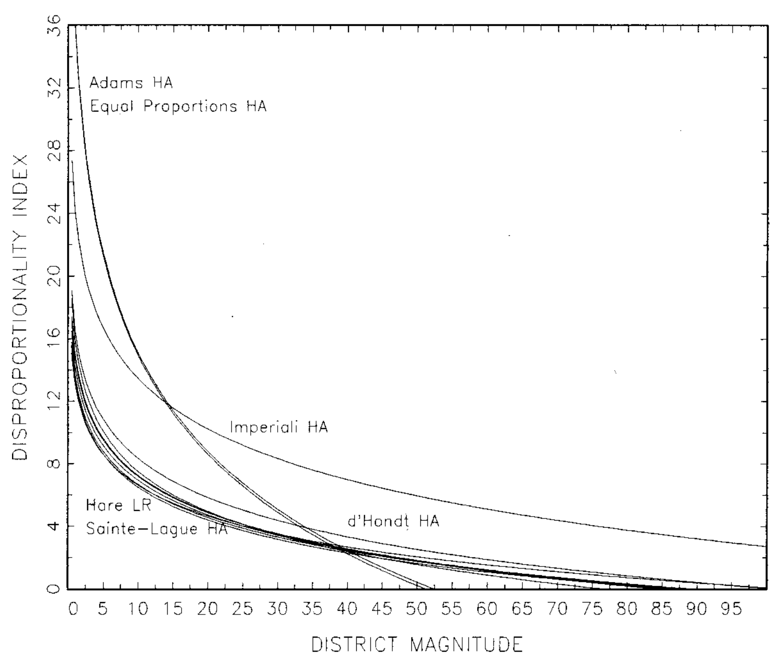

Fig. 1 Least-squares disproportionality as a function of district magnitude.

Table 3 District magnitude vs formula: Predictions of least-squares disproportionality

	District magnitude			
Formula	$M = 5$	$M = 10$	$M = 15$	$M = 20$
Imperiali HA	16.67	13.44	11.55	10.21
D'Hondt HA	10.81	8.32	6.86	5.82
Droop LR	8.77	6.66	5.42	4.54
Hare LR	8.57	6.60	5.45	4.64
St.-Laguë	8.57	6.60	5.45	4.64

a function of district magnitude is a feature we should consider independently for each formula.[3]

The exception with regard to previous rankings concerns the classic Sainte-Laguë formula. The results here suggest that this formula generates more proportional results than previously expected. The Sainte-Laguë HA method may yield more proportional results than the LR Hare and may consistently yield more proportional results than the LR Droop.[4] For example, consider the $M = 7$ list allocation drawn from the town of Sárbogárd, whose votes were $V = \{397, 394, 285, 224, 209, 172, 136\}$. Both the LR Hare and the LR Droop yield seats $S = \{2, 1, 1, 1, 1, 1, 0\}$, producing a least-squares disproportionality score of 9.86. Sainte-Laguë, on the other hand, awards one seat to each party, producing a least-squares disproportionality score of 9.83. Nonethless, these are minor differences; in the majority of cases the LR Hare and Sainte-Laguë produced identical results. The LR Droop, with its lower quota, tended to produce more disproportional results by leaving fewer remainder seats. One of many examples occurred in the city of Gödölő, with its nine list seats to be allocated. The votes were $V = \{2374, 1274, 245, 230\}$, yielding distributions of $S = \{6, 3, 0, 0\}$ for the LR Droop but $S = \{5, 3, 1, 0\}$ for both the LR Hare and the Sainte-Laguë. This made a difference in disproportionality of 8.80 for the LR Droop and 5.83 for the LR Hare and Sainte-Laguë, suggesting that finer distinctions should be drawn when ranking the Sainte-Laguë next to remainder methods as a single category. Other tests and numerous explorations of specific examples lead to the conclusion that the Sainte-Laguë and LR Hare produce essentially similar results and that the Sainte-Laguë is more proportional than all of the largest remainder methods except the LR Hare, particularly when $M < 10$.

Perhaps the most important result concerns the relative significance of electoral formula versus district magnitude. A comparison of fitted values at common ranges of district magnitude (Table 3) indicates that for common values of district magnitude—within the ranges identified in Section 2 as being relevant in the world's electoral systems—the difference between two formulas may be greater than the difference caused by a change in district magnitude. For instance, at $M = 10$, the difference is slightly greater when changing from the d'Hondt to the Droop than when increasing the district magnitude by half again, to 15.

[3]The results of other analyses not presented in Table 2 revealed only very minor differences even when using alternative proportionality indexes, such as the Loosemore–Hanby index. This is true both generally and specifically with regard to the Sainte-Laguë results discussed below, even when using Gallagher's (1991) Sainte-Laguë disproportionality index.

[4]The difference between coefficients is also statistically significant at the $p < .001$ levels, when standard tests for the equality of linear coefficients (Greene 1993, p. 187) are applied to both the intercept and the slope coefficients for the Sainte-Laguë, Hare, and Droop formulas. Exact results are provided in the replication data set.

Generally the results indicate that a shift from the d'Hondt to the Droop formula is at least as significant in affecting proportionality as changing M by 5 seats. And when the Imperiali HA formula is compared to any other method, formula type causes a much greater difference than changing district size. These results reinforce and quantify Taagepera and Laasko's (1980, p. 443) finding that "in the middle range of district magnitude seat distribution rules tend to carry some weight," especially given that the world's electoral systems tend to cluster precisely around "middle-range" values between $M = 5$ and $M = 15$. Yet the results challenge the corollary notion—at least within the range of district magnitude identified here—that "the number of seats allocated in an electoral district has a stronger impact on proportionality than almost any other factor, such as the choice between Sainte-Laguë or d'Hondt allocation formulas" (Taagepera and Shugart, 1989, p. 112). The inferential results presented here suggest that at between $5 \leq M \leq 15$ this proposition should be replaced by a more nuanced understanding of the relative determinants of proportionality.

4 The Advantage to Large Parties

How electoral formulas advantage the largest party is a second dimension according to which formulas may be compared. Table 4 presents the results of a regression without district magnitude, since tests including this variable showed no relationship between district size and the variable BONUSRAT for any of the electoral formulas. The coefficients therefore represent an average BONUSRAT for each formula from the data. The results produce a reordering of the ranking by Gallagher (1992), although they generally agree for the most commonly used formulas. Imperiali formulas in both HA and LR forms are shown to favor large parties more than the d'Hondt or any of the Sainte-Laguë variants. The Hare formula, interestingly, favored large parties less in the sample than did the Adams formula, which Gallagher ranked as the least favorable toward large parties. The regression results, in contrast, clearly show the Danish HA formula as the least favorable toward large parties. The analysis also equates (as did Gallagher's) the favorability toward the largest party of the Hare LR and the Sainte-Laguë methods. The more precise results presented here, based on more comprehensive data, nonetheless suggest that Gallagher's ranking may need fine-tuning.

Table 4 Comparing the favorability to large parties of different electoral formulas[a]

Formula	*Combined regression coefficient*	*Gallagher (1992) ranking*	
Plurality	2.90		*Most favorable*
Imperiali HA	1.44	Imperiali HA	↑
d'Hondt HA	1.20	Imperiali LR	
Imperiali LR	1.12	d'Hondt HA	
Hungarian St.-Laguë HA	1.12		
Modified St.-Laguë HA	1.10	Droop LR	
Droop LR	1.08	Modified St.-Laguë HA	
Equal proportions HA	1.08	Hare LR/St.-Laguë HA	
Adams HA	1.05	Equal proportions HA	
Hare LR	1.04		
St.-Laguë HA	1.04	Danish HA	↓
Danish HA	0.98	Adams HA	*Least favorable*

[a]Dependent variable: BONUSRAT. $n = 4274$, $\sigma = .62$, $R^2 = .68$. All coeffcients significant at the $p < .0001$ level.

A comparison of the ranking in Table 4 with that in Table 2 indicates that the least proportional formulas are not necessarily the ones that overrepresent the largest party most. Formulas that overrepresent small parties, such as the Adams and equal proportions methods, may score high on disproportionality but low on advantageousness toward large parties. Furthermore, the most proportional formulas, such as the Droop and Hare LR and Sainte-Laguë HA, are in the middle range of bonus to large parties. The conclusion is that although the distinction is not always made, there is indeed a difference between the proportionality of an electoral formula and its favorability toward the largest party.

5 Concluding Remarks

This reexamination of the proportionality profile debate in electoral systems research should lead to both a reinforcement and a slight reevaluation of previous findings. First, the more precise empirical approach and more explicit parameterization developed here generally confirm existing rankings of proportionality, with the exception of the classic Sainte-Laguë method. Not only is the Sainte-Laguë method shown to be the most proportional method of the 11 formulas considered, but in practice it yields results nearly equivalent to those with the Hare and Droop LR methods, previously considered the most proportional. Its treatment of large parties is also shown to be identical to that of the Hare LR method. Second, the results underscore the difference between a formula's favorability toward large parties and its disproportionality. Disproportionality may arise either from overrepresenting large parties and underrepresenting small ones or from underrepresenting larger parties and overrepresenting small ones. These two types of disproportionality have distinctly different political purposes and consequences. Finally, the separation of the influence of district magnitude relative to formula indicates that a difference in electoral formula such as the switch from the Sainte-Laguë to the d'Hondt may be at least as important as district magnitude in determining the proportionality of legislative representation. Together these results should remind researchers and electoral system designers alike that electoral formulas, even apparent minutiae such as quota denominators and numerical series, are more than inconsequential mathematical details.

References

Amorim-Neto, Octavio, and Gary W. Cox. 1997. "Electoral Institutions, Cleavage Structures, and the Number of Parties." *American Journal of Political Science* 41.

Benoit, Kenneth. 2000. "District Magnitude, Electoral Formula, and the Number of Parties." *European Journal of Political Research* (in press).

Blondel, J. 1969. *An Introduction to Comparative Government*. London: Weidenfeld and Nicholson.

Cox, Gary. 1997. *Making Votes Count: Strategic Coordination in the World's Electoral Systems.* Cambridge, UK: Cambridge University Press.

Gallagher, Michael. 1991. "Proportionality, Disproportionality, and Electoral Systems." *Electoral Studies* 10:33–51.

Gallagher, Michael. 1992. "Comparing Proportional Representation Electoral Systems: Quotas, Thresholds, Paradoxes and Majorities." *British Journal of Political Science* 22:469–496.

Greene, William H. 1993. *Econometric Analysis*, 2nd ed. New York: Macmillan.

Lijphart, Arend. 1986. "Degrees of Proportionality of Proportional Representation Formulas." In *Electoral Laws and Their Political Consequences*, ed. Bernard Grofman and Arend Lijphart. New York: Agathon Press.

Lijphart, Arend. 1994. *A Study of Twenty-Seven Democracies 1945–1990*. New York: Oxford University Press.

Loosemore, John, and Vincent Hanby. 1971. "The Theoretical Limits of Maximum Distortion: Some Analytic Expressions for Electoral Systems." *British Journal of Political Science* 1:467–477.

Taagepera, Rein, and Markku Laasko. 1980. "Proportionality Profiles of West European Electoral Systems." *European Journal of Political Research* 8:423–446.

Taagepera, Rein, and Matthew S. Shugart. 1989. *Seats and Votes: The Effects and Determinants of Electoral Systems*. New Haven, CT: Yale University Press.