# On the Robustness of Majority Rule*

by

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**Abstract**

We show that simple majority rule satisfies five standard and attractive axioms—the Pareto property, anonymity, neutrality, independence of irrelevant alternatives and (generic) decisiveness—on a bigger class of preference domains than (essentially) any other voting rule. Hence, in this sense, it is the most *robust* voting rule. This characterization of majority rule provides both an alternative to and generalization of May's (1952) characterization.

## 1. Introduction

How should a society select a president? How should a legislature decide which version of a bill to enact?

The casual response to these questions is probably to recommend that a vote be taken. But there are many possible voting rules—majority rule, plurality rule, rank-order voting, unanimity rule, approval voting, instant runoff voting, and a host of others (a *voting rule*, in general, is any method for choosing a winner from a set of candidates on the basis of voters' reported preferences for those candidates[1])—and so this response by itself, does not resolve the question. Accordingly, the theory of voting typically attempts to evaluate voting rules systematically by examining which fundamental properties or *axioms* they satisfy.

One generally accepted axiom is the *Pareto property*, the principle that if all voters prefer candidate $x$ to candidate $y$, then $x$ should be chosen over $y$.[2] A second axiom with strong appeal is *anonymity*, the notion that no voter should have more influence on the outcome of an election than any other[3](sometimes called the "one person/one vote" principle). Just as anonymity demands that all *voters* be treated alike, a third principle,

[1] In many electoral systems, a voter reports only his or her favorite candidate, rather than express a ranking of all candidates. If there are just two candidates (as in referenda, where the "candidates" are typically "yes" and "no"), then both sorts of reports amount to the same thing. But with three or more alternatives, knowing just voters' favorite is not enough to conduct some of the most prominent voting methods, such as majority rule and rank-order voting.

[2] Although the Pareto property is quite uncontroversial in the context of political elections, it is not so readily accepted—at least by noneconomists (philosophers, in particular)—in other social choice settings. Suppose, for example, that the "candidates" were two different national health care plans. Then, we might well imagine that factors such as fairness, scope of choice, degree of centralization could to some degree supplant citizens' preferences.

[3] Like the Pareto property, anonymity is not so widely endorsed in nonelection settings. In our healthcare scenario (see footnote 2), for example, it might be considered proper to give more weight to citizens with low incomes.

*neutrality*, requires the same thing for *candidates*: no candidate should get special treatment.[4]

Two particularly prominent voting rules that satisfy all three axioms—Pareto, anonymity, and neutrality—are (i) *simple majority rule*, according to which candidate $x$ is chosen if, for all other candidates $y$ in the available set, more voters prefer $x$ to $y$ than $y$ to $x$; and (ii) *rank-order voting* (also called the Borda count[5]), under which each candidate gets one point for every voter who ranks her first, two points for every voter who ranks her second, etc., and candidate $x$ is chosen if his point total is lowest among those in the available set.

But rank-order voting fails to satisfy a fourth standard principle, *independence of irrelevant alternatives* (IIA), which has attracted considerable attention since its emphasis in Nash (1950) and Arrow (1951).[6] IIA dictates if candidate $x$ is chosen from the set, and now some other candidate $y$ is removed from the set, then $x$ is still chosen..[7] To see why rank-order voting violates IIA consider an electorate consisting of five voters, Ann, Bob, Charlie, Doris, and Elsie. Suppose that there are three candidates—$x$, $y$, and $z$—and that Ann, Bob, and Charlie all prefer $x$ to $y$ and $y$ to $z$ but that Doris and Elsie both prefer $y$ to $z$ and $z$ to $x$. Then, $y$ will win the election with a point total of 8 (2 points from two first-

[4] Neutrality is hard to quarrel with in the setting of political elections. But if instead the "candidates" are, say, various amendments to a nation's constitution, then one might want to give special treatment to the alternative corresponding to *no* change—i.e., the status quo—to ensure that the constitutional change occurs only with overwhelming support.

[5] After the eighteenth-century French engineer Jean-Charles Borda, who first formalized rank-order voting rules.

[6] The Nash and Arrow versions of IIA differ somewhat. Here we follow the Nash formulation.

[7] IIA – although perhaps not quite so transparently desirable as the other three—has at least two strong arguments in its favor. First, as the name implies, it ensures that the outcome of an election will be unaffected by whether candidates who themselves have no chance of winning are on the ballot or not. Second, it is closely connected with the property that voters should have no incentive to vote *strategically*, i.e., at variance with their true preferences (see Theorem 4.73 in Dasgupta, Hammond, and Maskin 1979). Still, it has generated considerably more controversy than the other properties, particularly among proponents of rank-order voting (the Borda count), which famously violates IIA (see the text to follow).

place votes and 6 points from three second–place votes), compared with point totals of 9 for *x* and 10 for *z*. But notice that if *z* is not a candidate at all, then *x* will win with a point total of 7 (3 points from three first-place votes and 4 points from two second-place votes) compared with *y*'s point total of 8. Thus, whether the "irrelevant" candidate *z* is present or absent determines the outcome under rank-order voting, contradicting IIA.

Under majority rule (we will henceforth omit the qualification "simple" when this does not cause confusion with other variants of majority rule), by contrast, the choice between *x* and any other candidate *y* turns on only how many voters prefer *x* to *y* and how many *y* to *x* - - and not on whether or not some third candidate *z* is an option. Thus, in the above example, *x* is the winner (it beats all other candidates in head-to-head comparisons) whether or not *z* is on the ballot. In other words, majority rule satisfies IIA.

But majority rule has a well-known flaw, discovered by Borda's arch rival the Marquis de Condorcet (1785) and illustrated by the Paradox of Voting (or Condorcet Paradox): it may fail to generate *any* winner. Specifically, suppose that there are three voters 1, 2, 3, three alternatives *x*, *y, z*, and that the profile of voters' preferences is as follows

| 1 | 2 | 3 |
|---|---|---|
| $x$ | $y$ | $z$ |
| $y$ | $z$ | $x$ |
| $z$ | $x$ | $y$ |

(i.e., voter 1 prefers *x* to *y* to *z*, voter 2 prefers *y* to *z* to *x*, and voter 3 prefers *z* to *x* to *y*). Then, as Condorcet noted, a two-thirds majority prefers *x* to *y*, so that *y* cannot be chosen; a majority prefers *y* to *z*, so that *z* cannot be chosen; and a majority prefers *z* to *x*, so that *x* cannot be chosen. That is, majority rule fails to select *any* alternative; it violates *decisiveness*, which requires that a voting rule pick a (unique) winner.

In view of the failure of these two prominent voting methods, rank-order voting and majority rule, to satisfy all of the five axioms—Pareto, anonymity, neutrality, IIA and decisiveness—it is natural to enquire whether there is some other voting rule that might succeed where they fail. Unfortunately, the answer is negative: *no* voting rule satisfies all five axioms when there are three or more candidates (see Theorem 1), a result closely related to Arrow's (1951) Impossibility Theorem.

Still, there is an important sense in which this conclusion is too pessimistic: it presumes that to satisfy an axiom a voting rule must conform to that axiom *regardless* of what the combination of voters' preferences turn out to be.[8] Yet, in practice, some combinations may be highly unlikely. One reason for this may be ideology. As Black (1948) noted, in many elections, the typical voter's attitudes toward the leading candidates will be governed largely by how far away they are from his own position in left-right ideological space. This means that in the 2000 U.S. presidential election, where the four major candidates from left to right were Ralph Nader, Al Gore, George W. Bush, and Pat Buchanan, a voter favoring Gore might have the ranking

| | | |
|---|---|---|
| Gore<br>Nader<br>Bush<br>Buchanan | or even | Gore<br>Bush<br>Nader<br>Buchanan, |

but would be highly unlikely to rank the candidates in say, the order

Gore
Buchanan
Bush
Nader,

[8] In technical language, this is called the *unrestricted domain* requirement.

since Bush is closer than Buchanan to Gore ideologically. In other words, the graph of a voter's utility for candidates will be *single-peaked* when the candidates are arranged ideologically on the horizontal axis. Single-peakedness is of interest because, as Black showed, majority rule satisfies decisiveness generically[9] when voters' preferences conform to this restriction.

In fact, single-peakedness is by no means the only plausible restriction on preferences that ensures the decisiveness of majority rule. The 2002 French presidential election, where the three main candidates were Lionel Jospin (Socialist), Jacques Chirac (Conservative), and Jean-Marie Le Pen (National Front), offers another example. In that election, voters—regardless of their views on Jospin and Chirac—had extreme views on Le Pen: polls suggested, that among the three candidates, nearly everybody ranked him first or last; very few placed him in between. Whether such polarization is good for France is open to debate, but it is definitely good for majority rule: as we will see in section 3, such a restriction—in which one candidate is never ranked second—guarantees, like single-peakedness, that majority rule will be generically decisive.

Thus, majority rule *works well*—in the sense of satisfying our five axioms—for some domains of voters' preferences, but not for others (including the unrestricted domain). A natural issue to raise, therefore, is how its performance compares with that of other voting rules. As we have already noted, no voting rule can work well for *all* domains. So the obvious question to ask is: which voting rule(s) works well for the *biggest* class of

[9] We clarify what we mean by "generic" decisiveness below.

domains (and, in particular, is there a voting rule that works well for a bigger class of domains than majority rule does)?[10]

We show that majority rule is the unique answer to this question. Specifically, we establish (see Theorem 2) that if a given voting rule F works well on a domain of preferences, then majority rule works well on that domain too. Conversely, if $F$ differs from majority rule[11], there exists some other domain on which majority rule works well but $F$ does not.

Thus majority rule is essentially *uniquely* the voting rule that works well on the most domains; it is, in this sense, the most *robust* voting rule.[12] Indeed, this gives us a characterization of majority rule (see Theorem 3) different from the classic one derived by May (1952). For the case of two alternatives, May showed that majority rule is the *unique* voting rule satisfying a weak version of decisiveness, anonymity, neutrality, and a fourth property, *positive responsiveness*.[13] Our Theorem 3 strengthens decisiveness, omits positive responsiveness and imposes Pareto and IIA to obtain an alternative characterization that applies not just to the two-alternative case but *generall*y.

Theorem 2 is related to a result obtained in Maskin (1995). Like May, Maskin imposed somewhat different axioms from ours. In particular, instead of decisiveness—which requires that there be a unique winner—he allows for the possibility of multiple

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[10] It is easy to exhibit voting rules that satisfy four out of our five properties on *all* domains of preferences. For example, *supermajority rules* such as $\frac{2}{3}$-majority rule (which chooses alternative $x$ over alternative $y$ if $x$ garners at least a two-third majority over $y$) satisfy Pareto, anonymity and neutrality, and IIA on *any* domain. Similarly, *rank-order* voting satisfies Pareto, anonymity, neutrality, and decisiveness on any domain.

[11] More accurately, the hypothesis is that $F$ differs from majority rule for a "regular" preference profile belonging to a domain on which majority rule works well.

[12] More precisely, any other maximally robust voting rule can differ from majority rule only for "irregular" profiles on any domain on which it works well (see Theorem 3).

[13] A voting rule is positively responsive if wherever alternative $x$ is chosen (perhaps not uniquely) for a given specification of voters' preferences and those preferences are then changed only so that $x$ moves up in some voter's ranking, then $x$ becomes uniquely chosen.

winners but insists on *transitivity* (indeed, the same is true of some earlier versions of this paper; see Dasgupta and Maskin 1998): if $x$ beats $y$ and $y$ beats $z$, then $x$ should beat $z$. But more significantly, his proposition requires two strong and somewhat unpalatable assumptions. The first is that the number of voters be odd. This is needed to rule out exact ties: situations where exactly half the population prefers $x$ to $y$ and the other half prefers $y$ to $x$ (oddness is also needed for much of the early work on majority rule, e.g., Inada, 1969). In fact, our own results also call for avoiding such ties. But rather than simply assume an odd number of voters, we use the fact that even when there is an even number, an exact tie is unlikely if there are many voters. Hence, we suppose a large number of voters and ask only for *generic* decisiveness (i.e., decisiveness for "almost all" preferences). Formally, we work with a *continuum* of voters, but it will become clear that we could alternatively assume a finite number by defining generic decisiveness to mean "decisive for a sufficiently high proportion of preferences." In this way, we avoid "oddness" (an unappealing assumption, since it presumably holds only half the time).

Second, Maskin (1995) invokes the restrictive assumption that the voting rule $F$ being compared with majority rule satisfies Pareto, anonymity, IIA and neutrality on *any* domain. This is quite restrictive because, although it accommodates certain methods (such as the supermajority rules and the Pareto-extension rules), it eliminates such voting rules as the Borda count, plurality voting, and runoff voting. These are the most common alternatives in practice to majority rule, yet fail to satisfy IIA on the unrestricted domain. We show that this assumption can be dropped altogether.

We proceed as follows. In section 2, we set up the model. In section 3, we define our five properties formally: Pareto, anonymity, neutrality, independence of irrelevant alternatives, and generic decisiveness. For completeness (Theorem 1), we show that no voting rule always satisfies these properties, i.e., always works well. In section 4 we establish two lemmas that characterize when rank-order voting and majority rule work well. We use the second lemma in section 5 to establish our main result, Theorem 2. We obtain our alternative to May's (1952) characterization as Theorem 3. Finally, in section 6 we discuss a few extensions.

## 2. The Model

Our model in most respects falls within a standard social-choice framework. Let $X$ be the set of social alternatives (including alternatives that may turn out to be infeasible). For technical convenience, we take $X$ to be finite with cardinality $m(\geq 2)$. The possibility of individual indifference often makes technical arguments in the social-choice literature a great deal messier (see, for example, Sen and Pattanaik, 1969). We shall simply rule it out by assuming that voters' preferences can be represented by *strict orderings* (of course, with only a finite number of alternatives, the assumption that a voter is not exactly indifferent between any two alternatives does not seem very strong.) If $R$ is a strict ordering, then, for any alternatives $x, y \in X$ with $x \neq y$, the notation $xRy$ denotes "$x$ is strictly preferred to $y$ in ordering $R$."[14] For any subject $Y \subseteq X$ and any strict ordering $R$, let $R|_Y$ be the restriction of $R$ to $Y$.

Let $\Re_X$ be the set of all logically possible strict orderings of $X$. We shall typically suppose that voters' preferences are drawn from some subset $\Re \subseteq \Re_X$. For example, for

[14] Formally, a strict ordering (sometimes called a "linear ordering") is a binary relation that is reflexive, complete, transitive, and antisymmetric (antisymmetry means that if $xRy$ and $x \neq y$, then it is not the case that $yRx$)

some ordering $(x_1, x_2, \ldots, x_m)$ of the social alternatives, $\Re$ consists of single peaked preferences (relative to this ordering) if, for all $R \in \Re$, whenever $x_i R x_{i+1}$ for some *i*, then $x_j R x_{j+1}$ for all $j > i$; and whenever $x_{i+1} R x_i$ for some *i*, then $x_{j+1} R x_j$ for all $j < i$.

For the reason mentioned in the Introduction (and elaborated on below), we shall suppose that there is a *continuum* of voters indexed by points in the unit interval $[0,1]$. A *profile* $\boldsymbol{R}$ on $\Re$ is a mapping

$$\boldsymbol{R} : [0,1] \to \Re ,$$

where $\boldsymbol{R}(i)$ is voter *i*'s preference ordering. Hence, profile $\boldsymbol{R}$ is a specification of the preferences of all voters. For any $Y \subseteq X$, $\boldsymbol{R}|_Y$ is the profile $\boldsymbol{R}$ restricted to *Y*.

We shall use Lebesgue measure $\mu$ as our measure of the size of voting blocs.[15] Given alternatives *x* and *y* and profile $\boldsymbol{R}$, let

$$q_{\boldsymbol{R}}(x, y) = \mu\{i \mid x\boldsymbol{R}(i)y\} .$$

Then $q_{\boldsymbol{R}}(x, y)$ is the fraction of the population preferring *x* to *y* in profile $\boldsymbol{R}$.

A *voting rule F* is a mapping that, for each profile $\boldsymbol{R}$ on $\Re_X$[16] and each feasible subset $Y \subseteq X$, assigns a subset (possibly empty) $F(\boldsymbol{R}, Y) \subseteq X$, where if $\boldsymbol{R}|_Y = \boldsymbol{R}'|_Y$, then

[15] Because Lebesgue measure is not defined for all subsets of $[0,1]$, we will restrict attention to profiles $\boldsymbol{R}$ such that, for all $x$ and $y$, $\{i \mid x\boldsymbol{R}(i)y\}$ is a Borel set. Call these *Borel profiles*.

[16] Strictly speaking, we must limit attention to Borel profiles—see footnote 15—but henceforth we will not explicitly state this qualification.

$F(\boldsymbol{R}, Y) = F(\boldsymbol{R}', Y)$.[17] *Y* can be interpreted as the feasible set of alternatives and $F(\boldsymbol{R}, Y)$ as the *winning candidate(s)*.

For example, suppose that $F^M$ is *simple majority rule*. Then, for all $\boldsymbol{R}$ and *Y,*

$$F^M(\boldsymbol{R}, Y) = \left\{ x \in Y \middle| q_{\boldsymbol{R}}(x, y) \geq q_{\boldsymbol{R}}(y, x) \text{ for all } y \in Y \right\},$$

i.e., *x* is a winner in *Y* provided that, for any other alternative $y \in Y$, the proportion of voters preferring *x* to *y* is no less than the proportion preferring *y* to *x.* Such an alternative x is called a *Condorcet winner*. Note that there may not always be a Condorcet winner, i.e., $F^M(\boldsymbol{R}, Y)$ need not be nonempty (if, for example, the profile corresponds to that in the Condorcet Paradox).

A second example comprises the "supermajority" rules. For instance, *two-thirds majority* rule $F^{2/3}$ can be defined so that, for all $\boldsymbol{R}$ and *Y*,

$$F^{2/3}(R, Y) = Y',$$

where $Y'$ is the biggest subset of *Y* such that, for all $x, y \in Y'$ and $z \in Y - Y'$, $q_{\boldsymbol{R}}(x, y) < \frac{2}{3}$ and $q_{\boldsymbol{R}}(x, z) \geq \frac{2}{3}$. That is, *x* is a winner if it beats all nonwinners by at least a two-thirds majority, and it neither beats nor is beaten by any other winner by two-thirds majority or more.

As a third example, consider *rank-order voting*. Given $R \in \Re_X$ and *Y*, let $v_R^Y(x)$ be 1 if *x* is the top-ranked alternative of *R* in *Y*, 2 if *x* is second-ranked in *Y*, and so on. Then, given profile $\boldsymbol{R}$, $\int_0^1 v_{\boldsymbol{R}(i)}^Y(x)\, d\mu(i)$ is alternative *x*'s rank-order score (the total

[17] The requirement that $F(\boldsymbol{R}, Y) = F(\boldsymbol{R}', Y)$ if $\boldsymbol{R}|_Y = \boldsymbol{R}'|_Y$ may seem to resemble IIA, but it is actually much weaker. It says merely that the winner(s) should be determined only by voters' preferences for feasible candidates, and not by their preferences for infeasible ones. Indeed, *all* the voting rules we have discussed—*including* rank-order voting—satisfy this requirement.

number of points assigned to *x*) or *Borda count*. If $F^{RO}$ is *rank-order voting*, then, for all $\boldsymbol{R}$ and *Y*,

$$F^{RO}\left(\boldsymbol{R},Y\right)=\left\{x\in Y \middle| \int_0^1 v_{\boldsymbol{R}(i)}^Y\left(x\right)d\mu\left(i\right)\leq\int_0^1 v_{\boldsymbol{R}(i)}^Y\left(y\right)d\mu\left(i\right),\text{ for all } y\in Y\right\}.$$

Finally, consider plurality rule $F^P$ defined so that for all $\boldsymbol{R}$ and *Y*,

$$F^{P}\left(\boldsymbol{R},Y\right)=\left\{x\in Y \middle| \mu\left\{i \middle| x\boldsymbol{R}\left(i\right)y \text{ for all } y\in Y-\left\{x\right\}\right\} \geq \mu\left\{i \middle| z\boldsymbol{R}\left(i\right)y \text{ for all } y\in Y-\left\{x\right\}\right\}\text{for all } z\in Y\right\}$$

That is, *x* is a plurality winner if it is top-ranked in *Y* for at least as many voters as any other alternative in *Y*.

## 3. The Properties

We are interested in five standard properties that one may wish a voting rule to satisfy.

*Pareto Property on* $\mathfrak{R}$: For all $\boldsymbol{R}$ on $\mathfrak{R}$ and all $x,y\in X$ with $x\neq y$, if, for all $i$, $x\boldsymbol{R}\left(i\right)y$, then, for all *Y*, $x\in Y$ implies $y\notin F\left(\boldsymbol{R},Y\right)$.

In words, the Pareto property requires that if all voters prefer *x* to *y*, then the voting rule should not choose *y* if *x* is feasible. Virtually all voting rules used in practice satisfy this property. In particular, majority rule and rank-order voting (as well as supermajority rules and plurality rule) satisfy it on the unrestricted domain $\mathfrak{R}_X$.

*Anonymity on* $\mathfrak{R}$: Suppose that $\pi:\left[0,1\right]\rightarrow\left[0,1\right]$ is a measure-preserving permutation of $\left[0,1\right]$ (by "measure-preserving" we mean that, for all Borel sets

$T \subset [0,1]$, $\mu(T) = \mu(\pi(T))$). If, for all $\boldsymbol{R}$ on $\Re$, $\boldsymbol{R}^{\pi}$ is the profile such that $\boldsymbol{R}^{\pi}(i) = \boldsymbol{R}(\pi(i))$ for all *i,* then, for all *Y*, $F(\boldsymbol{R}^{\pi}, y) = F(\boldsymbol{R}, y)$.

In words, anonymity says that the winner(s) should depend only on the *distribution* of voters' preferences and not on *who* has those preferences. Thus if we permute the assignment of voters' preferences by $\pi$, the winners should remain the same. (The reason for requiring that $\pi$ be measure-preserving is to ensure that the fraction of voters preferring $x$ to $y$ be the same for $\boldsymbol{R}^{\pi}$ as it is for $\boldsymbol{R}$.) Anonymity embodies the principle that everybody's vote should count equally.[18] It is obviously satisfied on $\Re_X$ by both majority rule and rank-order voting, as well as by all other voting rules we have discussed so far.

*Neutrality on* $\Re$: Suppose that $\rho : X \to X$ is a permutation of $X$. For all $\boldsymbol{R}$ and *Y*, let $\boldsymbol{R}^{\rho,Y}$ be a profile such that, for all $i$ and all $x, y \in Y, x\boldsymbol{R}(i)y$ if and only if $\rho(x)\boldsymbol{R}^{\rho,Y}(i)\rho(y)$. Also, for all *Y*, let $Y^{\rho} = \{x \mid \rho^{-1}(x) \in Y\}$. Then, for all $\boldsymbol{R}$ on $\Re$ and all *Y*, $\rho(F(\boldsymbol{R}, Y)) = F(\boldsymbol{R}^{\rho,Y}, Y^{\rho})$, provided that there exists a profile $\boldsymbol{R}^{\rho,Y}$ on $\Re$.

In words, neutrality requires that a voting rule treat all alternatives symmetrically. Once again, all the voting rules we have talked about satisfy neutrality, including majority rule and rank-order voting.

As noted in the Introduction, we will invoke the Nash (1950) version of IIA:

*Independence of Irrelevant Alternatives* (IIA) *on* $\Re$: For all profiles $\boldsymbol{R}$ on $\Re$ and all *Y*, if $x \in F(\boldsymbol{R}, Y)$ and $Y'$ is a subset of *Y* such that $x \in Y'$, then $x \in F(\boldsymbol{R}, Y')$.

[18] Indeed, it is sometimes called "voter equality" (see Dahl, 1989).

Clearly, majority rule satisfies IIA on the unrestricted domain $\Re_X$ because if $x$ beats each other alternative by a majority, it continues to do so if some of those other alternatives are removed. However, rank-order voting violates IIA on $\Re_X$ as we showed by example.

Finally, we require that voting rules select a single winner:

*Decisiveness*: For all $\boldsymbol{R}$ and $Y$, $F(\boldsymbol{R}, Y)$ is a *singleton*, i.e., it consists of a single element.

Actually, decisiveness is too strong because, for example, it rules out the possibility of exact ties. Suppose, say, that $m = 2$ and exactly half the population prefers $x$ to $y$, while the other half prefers $y$ to $x$. Then no neutral voting rule will be able to choose between $x$ and $y$; they are perfectly symmetric. Nevertheless this indecisiveness is a knife-edge phenomenon - - it requires that the population be split precisely 50-50.

Thus, there is good reason for us to "overlook" it as pathological or *irregular*. And, because we are working with a continuum of voters, there is a formal way in which we can do so.

Specifically, let $S$ be a subset of (0, 1). A profile $\boldsymbol{R}$ on $\Re$ is *regular* with respect to $S$ (which we call an *exceptional set*) if, for all alternatives $x$ and $y$,

$$q_{\boldsymbol{R}}(x, y) \notin S.$$

That is, a regular profile is one for which the proportions of voters preferring one alternative to another all fall outside the specified exceptional set.

*Generic Decisiveness on* $\Re$: There exists a *finite* exceptional set $S$ such that, for all $Y$ and all profiles $\boldsymbol{R}$ on $\Re$ that are regular with respect to $S$, $F(\boldsymbol{R}, Y)$ is a singleton.

In other words, generic decisiveness requires that a voting rule be decisive only for regular profiles, ones where the preference proportions do not fall into some finite exceptional set. For example, as Lemma 2 below implies, majority rule is generically decisive on a domain of single-peaked preferences because there exists a unique winner for all regular profiles if the exceptional set consists of the single point $\frac{1}{2}$, i.e., $S = \left\{\frac{1}{2}\right\}$

In view of the Condorcet paradox, majority rule is not generically decisive on domain $\Re_X$. By contrast, rank-order voting is generically decisive on all domains including $\Re_X$. We shall say that a voting rule *works well* on a domain $\Re$ if it satisfies the Pareto property, anonymity, neutrality, IIA, and generic decisiveness on that domain. Thus, in view of our previous discussion, majority rule works well, for example, on a domain of single-peaked preferences. In section 4, we will give general characterization of when both majority rule and rank-order voting work well.

Although decisiveness is the only axiom for which we are considering a generic version, we could easily accommodate generic relaxations of the other conditions too. This seems pointless, however, because, to our knowledge, no commonly-used voting rule has nongeneric failures except with respect to decisiveness.

**Theorem 1:** If $m \geq 3$ no voting rule works well on $\Re_X$.

*Proof*: Suppose, contrary to the claim, *F* works well on $\Re_X$. We will use *F* to construct a social welfare function satisfying the Pareto property, anonymity, and IIA, contradicting the Arrow impossibility theorem (Arrow 1951).

Let *S* be the exceptional set for *F* on $\Re$. Because *S* is finite (by definition of generic decisiveness), we can find an integer $n \geq 2$ such that, if we divide the population into *n* equal groups, any profile for which all voters within a given group have the same ranking must be regular with respect to *S*. Let $\left[0,\frac{1}{n}\right]$ be group 1, $\left(\frac{1}{n},\frac{2}{n}\right]$ be group 2, …, and $\left(\frac{n-1}{n},1\right]$ be group *n*. Given profile $\boldsymbol{R}$ for which all voters within a given group have the same ranking and $X' \subseteq X$, let $\boldsymbol{R}^{X'}$ be the profile that is the same as $\boldsymbol{R}$ except that the elements of $X'$ have been moved to the top of all voters' rankings: for all *i*,

$$x\boldsymbol{R}^{X'}(i)\,y \text{ if and only if } x\boldsymbol{R}(i)\,y, \text{ for all } x,y \in X \text{ or } x,y \notin X'$$

$$x\boldsymbol{R}^{X'}(i)\,y, \text{ for all } x \in X' \text{ and } y \notin X'$$

Construct a social welfare function $f:\mathfrak{R}_X^n \to \mathfrak{R}_X$ so that, for all *n*-tuples $(R_1,\ldots,R_n) \in \mathfrak{R}_X^n$ and $x,y \in X$,

(1) $$xf(R_1,\ldots,R_n)\,y \text{ if and only if } x \in F\left(\boldsymbol{R}^{\{x,y\}},X\right),$$

where $\boldsymbol{R}$ corresponds to $(R_1,\ldots,R_n)$, i.e., it is the profile such that, for all *i* and *j*, $\boldsymbol{R}(i)=R_j$ if and only if voter *i* belongs to group *j*. To begin with, *f* is well-defined because, since *F* satisfies the Pareto principle, either $x \in F\left(\boldsymbol{R}^{\{x,y\}},X\right)$ or $y \in F\left(\boldsymbol{R}^{\{x,y\}},X\right)$. Similarly, *f* satisfies the Pareto principle and anonymity.[19] To see that *f* satisfies IIA (the Arrow 1951 version), consider two *n*-tuples $(R_1,\ldots,R_n)$ and $(\hat{R}_1,\ldots,\hat{R}_n)$ such that $(R_1,\ldots,R_n)\big|_{\{x,y\}} = (\hat{R}_1,\ldots,\hat{R}_n)\big|_{\{x,y\}}$, and let $\boldsymbol{R}$ and $\hat{\boldsymbol{R}}$ be the corresponding profiles. From generic decisiveness, Pareto, and IIA

(2) $$F\left(\boldsymbol{R}^{\{x,y\}},X\right) = F\left(\boldsymbol{R}^{\{x,y\}},\{x,y\}\right) \in \{x,y\}$$

[19] We have previously defined the Pareto property and anonymity for voting rules. Here we mean their natural counterparts for social welfare functions. So Pareto requires that if everyone prefers *x* to *y*, the social ranking will prefer *x* to *y*. And anonymity dictates that if we permute voters' rankings, the social ranking remains the same.

$$F\left(\hat{\boldsymbol{R}}^{\{x,y\}}, X\right) = F\left(\hat{\boldsymbol{R}}^{\{x,y\}}, \{x,y\}\right) \in \{x,y\} \quad .$$

But by definition of a voting rule, $F\left(\boldsymbol{R}^{\{x,y\}}, \{x,y\}\right) = F\left(\hat{\boldsymbol{R}}^{\{x,y\}}, \{x,y\}\right)$, and so from (1) and (3)

$$xf\left(R_1,\ldots,R_n\right)y \text{ and only if } xf\left(\hat{R}_1,\ldots,\hat{R}_n\right)y \quad ,$$

establishing Arrow-IIA.

Finally, we must show that $f$ is transitive. Consider an $n$-tuple $\left(R_1,\ldots,R_n\right)$ and alternatives $x$, $y$, $z$ for which

$$xf\left(R_1,\ldots,R_n\right)y \text{ and } yf\left(R_1,\ldots,R_n\right)z.$$

Consider $F\left(\boldsymbol{R}^{\{x,y,z\}}, X\right)$, where $\boldsymbol{R}$ is the profile corresponding to $\left(R_1,\ldots,R_n\right)$. From Pareto, $F\left(\boldsymbol{R}^{\{x,y,z\}}, X\right) \in \{x,y,z\}$. If $F\left(\boldsymbol{R}^{\{x,y,z\}}, X\right) = y$, then from IIA $F\left(\boldsymbol{R}^{\{x,y\}}, X\right) = y$, contradicting $xf\left(R_1,\ldots,R_n\right)y$. If $F\left(\boldsymbol{R}^{\{x,y,z\}}, X\right) = z$, we can derive a similar contradiction with $yF\left(R_1,\ldots,R_n\right)z$ Hence, $F\left(\boldsymbol{R}^{\{x,y,z\}}, X\right) = x$, and so we conclude, from IIA, that $F\left(\boldsymbol{R}^{\{x,z\}}, X\right) = x$, implying that $xF\left(R_1,\ldots,R_n\right)z$. Thus, transitivity obtains, and so $f$ is a social welfare function satisfying Pareto, anonymity, and IIA.

**Q.E.D.**

## 4. Preliminary Results

We have seen that rank-order voting violates IIA on $\Re_X$. We next show, however, that it satisfies IIA on domains for which "quasi-agreement" holds.

*Quasi-agreement on* $\Re$: Within each triple $\{x, y, z\} \subseteq X$, there exists an alternative, say $x$, such that either (a) for all $R \in \Re, xRy$ and $xRz$; or (b) for all $R \in \Re, yRx$ and $zRx$; or (c) for all $R \in \Re$, either $yRxRz$ or $zRxRy$.

In other words, quasi-agreement holds on domain $\Re$ if, for any triple of alternatives, all voters with preferences in $\Re$ agree on the relative ranking of *one* of these alternatives: either it is best within the triple, or it is worst, or it is in the middle.

**Lemma 1**: $F^{RO}$ satisfies IIA on $\Re$ if and only if quasi-agreement holds on $\Re$.

Proof**:** See appendix.

We turn next to majority rule. We already suggested in the previous section that a single-peaked domain ensures generic decisiveness. And we noted in the Introduction that the same is true when the domain satisfies the property that, for every triple of alternatives, there is one that is never "in the middle." But these are only sufficient conditions for generic transitivity; what we want is a condition that is both sufficient *and* necessary.

To obtain that condition, note that, for any three alternatives $x$, $y$, $z$, there are six logically possible strict orderings, which can be sorted into two Condorcet "cycles"[20]:

$$\begin{array}{ccc|ccc} x & y & z & x & z & y \\ y & z & x & z & y & x \\ z & x & y & y & x & z \end{array}$$

cycle 1 cycle 2

We shall say that a domain $\Re$ satisfies the *no-Condorcet-cycle* property [21] if it contains no Condorcet cycles. That is, for every triple of alternatives, at least one ordering is missing

[20] We call these *Condorcet* cycles because they constitute preferences that give rise to the Condorcet paradox

[21] Sen (1966) introduces this condition and calls it *value restriction*.

from each of cycles 1 and 2 (more precisely for each triple $\{x, y, z\}$, there do not exist orderings $R, R', R''$ in $\Re$ that, when restricted to $\{x, y, z\}$, generate cycle 1 or cycle 2).

**Lemma 2**: Majority rule is generically decisive on domain $\Re$ if and only if $\Re$ satisfies the no-Condorcet-cycle property.[22]

Proof**:** If there existed a Condorcet cycle for alternatives $\{x, y, z\}$ in $\Re$, then we could reproduce the Condorcet paradox by taking $Y = \{x, y, z\}$. Hence, the no-Condorcet-cycle property is clearly necessary.

To show that it is also sufficient, we must demonstrate, in effect, that the Condorcet paradox is the *only* thing that can interfere with majority rule's generic decisiveness. To do this, let us suppose that $F^M$ is not generically decisive on domain $\Re$. Then, in particular, if we let $S = \{\frac{1}{2}\}$ there must exist $Y$ and profile $\boldsymbol{R}$ on $\Re$ that is regular with respect to $\{\frac{1}{2}\}$ but for which $F^M(\boldsymbol{R})$ is either empty or contains multiple alternatives. If there exist $x, y \in F^M(\boldsymbol{R}, y)$, then

$$q_{\boldsymbol{R}}(x, y) = q_{\boldsymbol{R}}(y, x) = \tfrac{1}{2} \quad ,$$

contradicting the fact that $\boldsymbol{R}$ is regular with respect to $\{\frac{1}{2}\}$. Hence $F^M(\boldsymbol{R}, Y)$ must be empty. Choose $x_1 \in Y$. Then, because $x_1 \notin F^M(\boldsymbol{R}, y)$, there exists $x_2 \in y$ such that

$$q_{\boldsymbol{R}}(x_2, x_1) > \tfrac{1}{2} \quad .$$

Similarly, because $x_2 \notin F^M(\boldsymbol{R}, y)$, there exists $x_3 \in Y$ such that

$$q_{\boldsymbol{R}}(x_3, x_2) > \tfrac{1}{2} \quad .$$

[22] For the case of an odd and finite number of voters, Inada (1969) establishes that the no-Condorcet-cycle property is necessary and sufficient for majority rule to be transitive.

Continuing in this way, we must eventually reach $x_t \in Y$ such that

(3) $$q_{\boldsymbol{R}}\left(x_t, x_{t-1}\right) > \tfrac{1}{2}$$

But there exists some $\tau < t$ such that

(4) $$q_{\boldsymbol{R}}\left(x_\tau, x_t\right) > \tfrac{1}{2} \quad .$$

If $t$ is the smallest index for which this can happen, then

(5) $$q_{\boldsymbol{R}}\left(x_{t-1}, x_\tau\right) > \tfrac{1}{2} \quad .$$

Combining (3) and (5), we conclude that there must be a positive fraction of voters in ***R*** who prefer $x_t$ to $x_{t-1}$ to $x_\tau$, i.e.,

(6) $$\begin{matrix} x_t \\ x_{t-1} \\ x_\tau \end{matrix} \in \Re \quad .$$[23]

By similar argument, it follows that

$$\begin{matrix} x_{t-1} \\ x_\tau \\ x_t \end{matrix} \;,\; \begin{matrix} x_\tau \\ x_t \\ x_{t-1} \end{matrix} \in \Re \quad .$$

Hence, $\Re$ violates the no-Condorcet-cycle property, as was to be shown.

**Q.E.D.**

It is easy to check that a domain of single-peaked preferences satisfies the no-Condorcet-cycle property. Hence, Lemma 2 implies that majority rule is generically decisive on such a domain. The same is true of the domain we considered in the Introduction in connection with the 2002 French presidential election.

**5. The Robustness of Majority Rule**

We can now state our main finding:

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[23] To be precise, formula (6) says that there exists an ordering in $R \in \Re$ such that $x_t R x_{t-1} R x_\tau$.

**Theorem 2:** Suppose that voting rule $F$ works well on domain $\mathfrak{R}$. Then, majority rule $F^M$ works well on $\mathfrak{R}$ too. Conversely, suppose that $F^M$ works well on domain $\mathfrak{R}^M$. Then, if there exists profile $\boldsymbol{R}^\circ$ on $\mathfrak{R}^M$, regular with respect to $F$'s exceptional set, such that

$$(7) \qquad F\left(\boldsymbol{R}^\circ\right) \neq F^m\left(\boldsymbol{R}^\circ\right) \quad ,$$

there exists a domain $\mathfrak{R}'$ on which $F^M$ works well, but $F$ does not.

**Remark:** Without the requirement that the profile $\boldsymbol{R}^\circ$ for which $F$ and $F^M$ differ belongs to a domain on which majority rule works well, the converse assertion above would be false. In particular, consider a voting rule that coincides with majority rule except for profiles that violate the no-Condorcet-cycle property. It is easy to see that such a rule works well on *any* domain for which majority rule does because it coincides with majority rule on such a domain.

<u>Proof</u>**:** Suppose first that $F$ works well on $\mathfrak{R}$. If, contrary to the theorem, $F^M$ does not work well on $\mathfrak{R}$, then, from Lemma 2, there exists a Condorcet cycle in $\mathfrak{R}$:

$$(8) \qquad \begin{matrix} x \\ y \\ z \end{matrix}, \begin{matrix} y \\ z \\ x \end{matrix}, \begin{matrix} z \\ x \\ y \end{matrix} \in \mathfrak{R} \quad ,$$

for some $x, y, z \in X$. Let $S$ be the exceptional set for $F$ on $\mathfrak{R}$. Because $S$ is finite (by definition of generic transitivity), we can find an integer $n$ such that, if we divide the population into $n$ equal groups, any profile for which all the voters in each particular group have the same ordering in $\mathfrak{R}$ must be regular with respect to $S$.

Let $\left[0, \frac{1}{n}\right]$ be group 1, $\left(\frac{1}{n}, \frac{2}{n}\right]$ be group 2, …, and $\left(\frac{n-1}{n}, 1\right]$ be group $n$. Consider a profile $\boldsymbol{R}_1$ on $\mathfrak{R}$ such that all voters in group 1 prefer $y$ to $x$ and all voters in the other groups prefer $x$ to $y$. That is, the profile is

(9) $$\begin{array}{cccc} 1 & 2 & \cdots & n \\ \hline y & x & & x \\ x & y & & y \end{array} \quad .$$

From (8), such a profile exists on $\mathfrak{R}$.

Because $F$ is generically decisive on $\mathfrak{R}$, there are two cases: either (*i*) $F\left(\boldsymbol{R}_1,\{x,y\}\right)=x$ or (*ii*) $F\left(\boldsymbol{R}_1,\{x,y\}\right)=y$ strictly preferred to *x* under $F\left(\boldsymbol{R}_1\right)$.

*Case* (*i*): $F\left(\boldsymbol{R}_1,\{x,y\}\right)=x$

Consider a profile $\boldsymbol{R}_1^*$ on $\mathfrak{R}$ in which all voters in group 1 prefer *x* to *y* to *z*; all voters in group 2 prefer *y* to *z* to *x*; and all voters in the remaining groups prefer *z* to *x* to *y*. That is, [24]

(10) $$\boldsymbol{R}_1^* = \begin{array}{ccccc} 1 & 2 & 3 & \cdots & n \\ \hline x & y & z & & z \\ y & z & x & & x \\ z & x & y & & y \end{array} \quad .$$

Notice that, in profile $\boldsymbol{R}_1^*$, voters in group 1 prefer *x* to *z* and that all other voters prefer *z* to *x*. Hence, neutrality and the case (*i*) hypothesis imply that

(11) $$F\left(\boldsymbol{R}_1^*,\{x,z\}\right)=z \quad .$$

Observe also that, in $\boldsymbol{R}_1^*$, voters in group 2 prefer *y* to *x* and all other voters prefer *x* to *y*. Hence from anonymity and neutrality and the case (*i*) hypothesis, we conclude that

(12) $$F\left(\boldsymbol{R}_1^*,\{x,y\}\right)=x \quad .$$

Now (11), (12), IIA, and generic decisiveness imply that

(13) $$F\left(\boldsymbol{R}_1^*,\{x,y,z\}\right)=z \quad .$$

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[24] This is not quite right because we are not specifying how voters rank alternatives other than *x*, *y*, and *z*. But from IIA, these other alternatives do not matter for the argument.

But (13) and IIA imply that

$$F\left(\boldsymbol{R}_1^*, \{y, z\}\right) = z \quad .$$

Hence, from neutrality, for any profile $\boldsymbol{R}_2$ on $\Re$ such that

$$(14) \qquad \begin{array}{ccccc} 1 & 2 & 3 & \cdots & n \\ \hline y & y & x & & x \\ x & x & y & & y \end{array} \quad ,$$

we must have,

$$(15) \qquad F\left(\boldsymbol{R}_2, \{x, y\}\right) = x \quad .$$

That is, we have shown that if $x$ is chosen over $y$ when just *one* out of $n$ groups prefers $y$ to $x$ (as in (9)), then $x$ is again chosen over $y$ when *two* groups out of $n$ prefer $y$ to $x$ (as in (14)).

Now choose $\boldsymbol{R}_2^*$ on $\Re$ so that

$$(16) \qquad \boldsymbol{R}_2^* = \begin{array}{cccccc} 1 & 2 & 3 & 4 & \cdots & n \\ \hline x & y & y & z & & z \\ y & z & z & x & & x \\ z & x & x & y & & y \end{array} \quad .$$

Arguing as above, we can use (14) – (16) to show that $x$ is chosen over $y$ if *three* groups out of $n$ prefer $y$ to $x$. Continuing iteratively, we conclude that $x$ is chosen over $y$ even if $n-1$ groups out of $n$ prefer $y$ to $x$, which, in view of neutrality, violates the case (*i*) hypothesis. Hence case (*i*) is impossible.

*Case* (*ii*): $F\left(\boldsymbol{R}_1, \{x, y\}\right) = y$

But from the case (*i*) argument, case (*ii*) leads to the same contradiction as before. We conclude that $F^M$ must work well on $\Re$ after all, as claimed.

For the converse, suppose that there exists domain $\Re^M$ on which $F^M$ works well. If $F$ does not work well on $\Re^M$ too, we can take $\Re' = \Re^M$ to complete the proof. Hence, assume that $F$ works well on $\Re^M$ with exceptional set $S$ and that there exists $Y, x$ and $y \in Y$, and regular profile $\boldsymbol{R}^\circ$ on $\Re^M$ such that $y = F\left(\boldsymbol{R}^\circ, Y\right) \neq F^m\left(\boldsymbol{R}^\circ, Y\right) = x$. Then, from IIA, there exist $\alpha \in \left(0,1\right)$ with $\alpha \notin S$ and

(17) $$1-\alpha > \alpha \ ,$$

and $q_{\boldsymbol{R}^\circ}\left(x, y\right) = 1-\alpha$ such that

$$F^M\left(\boldsymbol{R}^\circ, \{x, y\}\right) = x \quad .$$

and

(18) $$F\left(\boldsymbol{R}^\circ, \{x, y\}\right) = y \quad .$$

Because $F$ satisfies IIA on $\Re^M$, we can assume that $\boldsymbol{R}^\circ$ consists of just two orderings $R', R'' \in \Re$ such that

(19) $$y\, R'x \text{ and } x\, R''y \quad .$$

Furthermore, because $F$ is anonymous on $\Re^M$, we can write $\boldsymbol{R}^\circ$ as

(20) $$\boldsymbol{R}^\circ = \frac{\left[0,\alpha\right)}{R'} \quad \frac{\left[\alpha,1\right]}{R''} \quad ,$$

so that voters between 0 and $\alpha$ have preferences $R'$, and those between $\alpha$ and 1 have $R''$.

Consider $z \notin \{x, y\}$ and profile $\boldsymbol{R}^{\circ\circ}$ such that

(21) $$\boldsymbol{R}^{\circ\circ} = \begin{array}{ccc} [0,\alpha) & [\alpha,1-\alpha) & [1-\alpha,1] \\ \hline z & z & x \\ y & x & z \\ x & y & y \end{array}.$$[25]

Note that in (21) we have left out the alternatives other than *x*, *y*, and *z*. To make matters simple, assume that the orderings of $\boldsymbol{R}^{\circ\circ}$ are the *same* for those alternatives. Suppose, furthermore, that, in these orderings, *x*, *y*, and *z* are each preferred to any alternative not in $\{x, y, z\}$. Then, because $\alpha \notin S, \boldsymbol{R}^{\circ\circ}$ is regular.

Let $\mathfrak{R}'$ consist of the orderings in $\boldsymbol{R}^{\circ\circ}$. From Lemma 2, $F^M$ works well on $\mathfrak{R}'$. So, we can assume that *F* does too (otherwise, we are done). From generic decisiveness and because $\boldsymbol{R}^{\circ\circ}$ is regular, $F\left(\boldsymbol{R}^{\circ\circ},\{x,y,z\}\right)$ is a singleton. From Pareto, we cannot have $F\left(\boldsymbol{R}^{\circ\circ},\{x,y,z\}\right)=y$, since *z* Pareto dominate *y*. If $F\left(\boldsymbol{R}^{\circ\circ},\{x,y,z\}\right)=x$, then from IIA $F\left(\boldsymbol{R}^{\circ\circ},\{x,y\}\right)=x$, contradicting (18). Thus, we must have $F\left(\boldsymbol{R}^{\circ\circ},\{x,y,z\}\right)=z$, implying from IIA that $F\left(\boldsymbol{R}^{\circ\circ},\{x,z\}\right)=z$, which contradicts (18) because of anonymity and neutrality. Hence, *F* does not work well on $\mathfrak{R}'$ after all.

**Q.E.D.**

As a simple illustration of Theorem 2, let us see how it applies to rank-order voting. If $X=\{x,y,z\}$, Lemma 1 implies that $F^{RO}$ works well, for example, on the domain

$$\left\{\begin{matrix} x \\ y \\ z \end{matrix}, \begin{matrix} z \\ y \\ x \end{matrix}\right\} .$$

[25] We have again left out the alternatives other than $x, y, z$, which we are entitled to do by IIA. To make matters simple, assume that the orderings of $\boldsymbol{R}^{\circ\circ}$ are all the *same* for these other alternatives. Suppose furthermore that, in these orderings, $x, y, z$ are each preferred to any alternative not in $\{x, y, z\}$.

And, as Theorem 2 guarantees, $F^M$ also works well on this domain, since it obviously does not contain a Condorcet cycle. Conversely, on the domain

(*) $$\Re' = \left\{ \begin{matrix} x \\ y \\ z \\ w \end{matrix}, \begin{matrix} y \\ z \\ x \\ w \end{matrix}, \begin{matrix} z \\ w \\ y \\ x \end{matrix} \right\},$$

$F^M\left(\boldsymbol{R},\{w,x,y,z\}\right) \neq F^{RO}\left(\boldsymbol{R},\{w,x,y,z\}\right)$ for the profile $\boldsymbol{R}$ in which the proportion of voters with ordering $\begin{matrix} x \\ y \\ z \\ w \end{matrix}$ is .3, the proportion with ordering $\begin{matrix} y \\ z \\ x \\ w \end{matrix}$ is .3 and that with ordering $\begin{matrix} z \\ w \\ y \\ x \end{matrix}$ is .4.

From Lemma 2, $F^M$ works well on $\Re'$ given by (*). Hence, from Lemma 1, $\Re'$ constitutes a domain on which $F^M$ works well but $F^{RO}$ does not, as guaranteed by Theorem 2.

In the Introduction, we mentioned May's (1952) characterization of majority rule for two alternatives. In view of Theorem 2, we can provide an alternative characterization, which also extends to more than two alternatives. Specifically, call two voting rules $F$ and $F'$ *generically identical* on domain $\Re$ if there exists a finite set $S \subset (0,1)$ such that $F(\boldsymbol{R},Y) = F'(\boldsymbol{R},Y)$ for all $Y$ and all $\boldsymbol{R}$ on $\Re$ for which $q_{\boldsymbol{R}}(x,y) \notin S$. Call $F$ *maximally robust* if there exists no other voting rule that (*i*) works well on every domain on which $F$ works well and (*ii*) works well on some domain on which $F$ does not work well. Theorem 1 implies that majority rule can be characterized as essentially the unique voting rule that satisfies Pareto, anonymity, neutrality, IIA and generic decisiveness on the most domains:

**Theorem 3:** Majority rule is maximally robust, and any other maximally robust voting rule *F* is generically the same as majority rule on any domain on which *F* or majority rule works well.

## 6. Further Work

The symmetry inherent in neutrality is often a reasonable and desirable property—we would presumably want to treat all candidates in a presidential election the same. However, there are also circumstances in which it is natural to favor certain alternatives. The rules for changing the U.S. Constitution are a case in point. They have been deliberately devised so that, at any time, the current version of the Constitution—the status quo—is difficult to revise.

In related work (Dasgupta and Maskin 2007), we show that when neutrality is *dropped* (and the requirement that ties be broken "consistently" is also imposed), then *unanimity rule with an order of precedence* [26](the rule according to which $x$ is chosen over $y$ if it precedes $y$ in the order of precedence, unless everybody prefers $y$ to $x$) supplants majority rule as the most robust voting rule. It is, of course, not surprising that with fewer axioms to satisfy, there should be voting rules that satisfy them all on a wider class of domains than majority rule does. Nevertheless, it is striking that, once again, the maximally robust rule is so simple and familiar.

[26] For discussion of this voting rule in a political setting see Buchanan and Tullock (1962).

Appendix

**Lemma 1**: For any domain $\Re$, $F^{RO}$ satisfies IIA on $\Re$ if and only QA holds on $\Re$.

Proof: Assume first that QA holds on $\Re$. We must show that $F^{RO}$ satisfies IIA on $\Re$. Consider profile $\boldsymbol{R}$ and subset $Y$ such that $F(\boldsymbol{R},Y)=x$

(A1) $$F(\boldsymbol{R},Y)=x \quad ,$$

for some $x\in X$. We must show that, for all $y\in Y-\{x\}$,

(A2) $$F(\boldsymbol{R},Y-\{y\})=x \quad .$$

Suppose, to the contrary that

(A3) $$F(\boldsymbol{R},Y-\{y\})=z,\ \text{for } z\in Y-\{x\} \quad .$$

(A1) and (A2) together imply that $z$ must rise relative to $x$ in some voters' rankings in $\boldsymbol{R}$ due to the deletion of $y$, i.e.

(A4) $$\begin{matrix} x \\ y \\ z \end{matrix} \in \Re \quad .$$

But (A2) implies that there exists $i$ such that

(A5) $$z\boldsymbol{R}(i)x \quad .$$

Hence (A4),(A5) and QA imply that

(A6) $$\left\{ \begin{matrix} x & z \\ y, & y \\ z & x \end{matrix} \right\} = \Re\big|_{\{x,y,z\}}$$

Now (A3) and (A6) imply that there exists $w\in Y$ such that (i) if $R\in\Re$ with $zRyRx$, then $zRwRx$ and (ii) if $R'\in\Re$ with $xR'yR'z$ then either (a) $wR'x$ or (b) $zR'w$. Thus, if (a) holds, we have

(A7) $$\begin{Bmatrix} z & w \\ w, & x \\ x & z \end{Bmatrix} \subseteq \Re|_{\{x,w,z\}}$$

and if (b) holds

(A8) $$\begin{Bmatrix} z & x \\ w, & z \\ x & w \end{Bmatrix} \subseteq \Re|_{\{x,w,z\}}$$

But (A7) and (A8) both violate QA< and so (A2) must hold after all.

Next, suppose that QA does not hold on $\Re$. Then there exist alternatives $x$, $y$, $z$ such that

(A9) $$\begin{Bmatrix} x & y \\ y, & z \\ z & x \end{Bmatrix} \subseteq \Re|_{\{x,y,z\}} \quad .$$

Consider the profile $\boldsymbol{R}^*$ in which proportion .6 of the population has ranking $\begin{matrix} x \\ y \\ z \end{matrix}$ and proportion .4 has $\begin{matrix} y \\ z \\ x \end{matrix}$. Then

$$F^{RO}\left(\boldsymbol{R}^*, \{x, y, z\}\right) = y \quad .$$

But,

$$F^{RO}\left(\boldsymbol{R}^*, \{x, y\}\right) = x \quad ,$$

Contradicting IIA, Thus a violation of QA implies that $F^{RO}$ does not work well.

**Q.E.D.**

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