Economic Theory **22**, 557–568 (2003)

Economic Theory

A strategy-proofness characterization of majority rule*

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Received: August 29, 2000; revised version: November 13, 2002

Summary. A feasible alternative x is a strong Condorcet winner if for every other feasible alternative y there is some majority coalition that prefers x to y. Let $\mathcal{L}_C$ (resp., $\wp_C$) denote the set of all profiles of linear (resp., merely asymmetric) individual preference relations for which a strong Condorcet winner exists. Majority rule is the only non-dictatorial and strategy-proof social choice rule with domain $\mathcal{L}_C$, and majority rule is the only strategy-proof rule with domain $\wp_C$.

Keywords and Phrases: Condorcet, Majority winner, Strategy-proof.

JEL Classification Numbers: D70, D71.

1 Introduction

Given a set X of alternatives, we can specify a *social welfare function* f that determines a binary relation on X as a function of individual preferences, or we can specify a *social choice rule* g that selects a single member of X as a function of individual preferences. May (1952) provided the first axiomatic characterization of majority rule as a social welfare function. May's characterization is based on Independence of Irrelevant Alternatives, Neutrality, Anonymity, and a strong positive responsiveness axiom. Maskin (1995) substituted the Pareto criterion for the last of these. He proved that any social welfare function satisfying the four axioms will fail to be transitive-valued at any individual preference profile at which majority rule violates transitivity, and, unless it is majority rule itself, will fail to be transitive-valued at some individual preference profile at which majority rule *is* transitive-valued. This was also established by Campbell and Kelly (2000) with a less demanding set of axioms.

* We are grateful to Wulf Gaertner and our two referees for insightful comments on a previous draft.
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The present paper characterizes the social choice rule g that selects the majority rule winner. Taking our cue from Maskin, we posit a large domain of profiles and then we show that majority rule is strategy-proof on that domain, and that no other non-dictatorial social choice rule is strategy proof on that domain. The largest conventional domain for which our proof is valid is the set of all n-tuples of linear orders at which there is a strong Condorcet winner – i.e., an alternative that defeats every other by a strict majority. Of course, we are loading the dice by employing a large domain compatible with the existence of a strong Condorcet winner, but doing so gives us a complement to Maskin's social welfare function characterization of majority rule. (We also assume an odd number of individuals. The conclusion touches briefly on the possibility of extending our result to even n, and to domains that allow individual indifference between distinct alternatives.)

We have a second strategy-proofness characterization of majority rule for finite X: If we enlarge the domain by adding profiles at which at least one of the individual relations has a three-term cycle at the top, but which still yield a strong Condorcet winner, then majority rule is the only strategy-proof social choice rule. (For infinite X, dictatorship is already ruled out within the set of all n-tuples of linear orders at which there is a strong Condorcet winner, because for every individual i there will be a profile in this set at which i's ordering does not have a maximal alternative.) Our interest in strategy-proofness under conditions that admit strict preference cycles in individual preference relations is two-fold: First, because intransitivities have shown up in laboratory experiments it is reasonable to ask how individual preference intransitivity bears on the preference revelation problem. [See Chapter 7 in Thaler (1992) for a discussion of the experiments. It is noteworthy that Saari (1994) employs intransitive individual preferences in his proof of Arrow's theorem.] Second, our result is a preliminary step in the investigation of domains on which there is one and only one strategy-proof social choice rule. This paper shows that one can identify domains on which majority rule is the unique strategy-proof rule. We also exhibit a domain on which the unique strategy-proof rule is dictatorial. (The domain of the Gibbard-Satterthwaite Theorem admits a different dictatorial rule for each individual.)

Given a profile of (not necessarily acyclic) individual preferences, a feasible outcome x is a *strong Condorcet winner* if for every other feasible alternative y the number of individuals who strictly prefer x to y exceeds the number of individuals who strictly prefer y to x. Let $\wp_C$ denote the set of profiles for which a strong Condorcet winner exists, and let g_C be the social choice rule that selects the Condorcet winner on $\wp_C$. It is easy to prove that no individual *or coalition* can manipulate g_C within $\wp_C$: Suppose that the profile p of individual preferences is such that x defeats every other alternative by a majority when everyone votes according to his true preference scheme. If alternative y ranks above x in the true preference ordering of everyone in coalition J, then x will defeat y by a majority, no matter how the individuals in J vote (provided that no one in $N \backslash J$ changes her reported preference relation), because x defeats y when everyone in J truthfully declares his preference for y over x. Therefore, the outcome cannot be y when the members of J misrepresent their preferences, and no one else does. We refer to this as the *folk theorem*. The logic of the folk theorem obviously applies to any domain that is a subset of $\wp_C$, but the argument depends crucially on the assumption that both the profile of true individual preferences *and* the profile of reported individual preferences belong to the family of profiles for which a strong Condorcet winner exists.

We have said that g_C is the only strategy-proof social choice rule with domain $\wp_C$. What about dictatorship? It is not *defined* at every profile in that set: For every individual i there will be profiles in $\wp_C$ at which i's preference scheme does not have a maximal alternative. (Suppose, for example, that $p(i)$ is any binary relation and there is an alternative x that is preferred to every other outcome for everyone other than i. Then x is a strong Condorcet winner, and the profile belongs to $\wp_C$.) Therefore, a social choice rule with domain $\wp_C$ cannot always selects i's most-preferred alternative. Our second theorem implies that dictatorship cannot be extended to $\wp_C$ without creating opportunities for advantageous misrepresentation.

Barberà, Massò, and Neme (1999) investigate the connection between strategy-proofness and the size of the domain from quite a different standpoint. They make the standard assumption that individual preferences are always transitive, and then characterize the maximal domains on which an arbitrary generalized median voter scheme is strategy-proof; these maximal domains have a generalized single-peakedness property. Berga (2002) proves a similar result in a continuous multidimensional framework, and discusses related results.

The next section contains basic definitions and examples. Section 3 proves our two main theorems, and Section 4 presents a simple dictatorship counterpart. The last section offers some concluding remarks.

2 Definitions and two examples

$N = \{1, 2, ..., n\}$ is the finite set of voters $(n > 1)$. A *majority coalition* is any subset of N containing at least $\frac{n+1}{2}$ individuals. X is the feasible set of alternatives. $A(X)$ denotes the set of *all* complete and antisymmetric binary relations on X: The relation $\succ$ belongs to $A(X)$ if and only if $x \neq y$ implies $x \succ y$ or $y \succ x$, but not both. We say that $\succ$ in $A(X)$ *contains a three-term cycle* if there are three alternatives x, y, and z such that $x \succ y \succ z \succ x$. Let $L(X)$ denote the set of transitive members of A(X) – i.e., the set of linear orderings on X.

When we use $(x, ...)$ to denote some member $\succ$ of $L(X)$ we are indicating that $x \succ y$ holds for all $y \in X\backslash\{x\}$. We say that x is the *maximal element* of $\succ$ in this case, and sometimes write $x =\succ [1]$. Similarly, $(x, y, ...)$ signifies that $x =\succ [1]$ and $y \succ z$ holds for all $z \in X\backslash\{x, y\}$, and we sometimes write $y =\succ [2]$ when $\succ$ is the relation $(x, y, ...)$. If $y \succ z$ for all $y \in X\backslash\{z\}$ we write $z =\succ [\infty]$, and we will use $(x, ..., z)$ to indicate that $x =\succ [1]$ and $z =\succ [\infty]$. Of course these alternatives will not exist for all members of $A(X)$, and not even for all members of $L(X)$ if X is infinite. But for infinite X and any finite positive integer m there will be members of $L(X)$ such that alternative $\succ [j]$ exists for $j = 1, 2, ..., m$, where we set $y =\succ [j]$ if $x \succ y$ holds for exactly $j - 1$ members of X and $y \succ z$ for all $z \in X\backslash\{\succ [1], \succ [2], ..., \succ [j-1]\}$.

A *profile* p is a function from N into $A(X)$, and $p(i)$ is the binary relation that p assigns to $i \in N$. For any $p \in A(X)^N$ we will let $\succ_i^p$ also denote the relation

$p(i)$, so that we can write $x \succ_i^p y$ if at profile p individual i strictly prefers x to y. For convenience, we let $\mathcal{L}$ denote the set $L(X)^N$. A (*resolute*) *social choice rule* $g : \wp \to X$ selects a member $g(p)$ of X for each $p \in \wp$. The *domain* $\wp$ of g is a subset of $A(X)^N$. The *range* of $g : \wp \to X$ is the set of $x \in X$ such that $g(p) = x$ for some $p \in \wp$. We say that g is *dictatorial* if there is some $i \in N$ such that for all $p \in \wp$ for which $p(i)$ has a maximal alternative, we have $p(i)[1] = g(p)$. We are ruling out individual indifference in this paper, and thus when an ordering has a maximal alternative it will be unique. However, in defining dictatorship we have taken into consideration the fact that we admit profiles for which there is no maximal alternative for some individual's preference ordering.

We say that x is a *strong Condorcet winner* at $p \in A(X)^N$ if $x \in X$ and if for every $y \in X\backslash\{x\}$ there is a majority coalition J such that $x \succ_i^p y$ for all $i \in J$. Let $\wp_C$ represent the set of all profiles $p \in A(X)^N$ such that there is a strong Condorcet winner at p. Let $\mathcal{L}_C = \mathcal{L} \cap \wp_C$ denote the set of all profiles of linear orders at which there is a strong Condorcet winner. We define the social choice rule g_C on $\wp_C$ by setting $g_C(p)$ equal to the strong Condorcet winner, for arbitrary $p \in \wp_C$. If $\wp \subset \wp_C$, and $\wp$ is the domain of g, and $g(p) = g_C(p)$ for all $p \in \wp$ we say that g is *majority rule*.

For any two profiles p and q in $A(X)^N$ define the *standard sequence* $\{r^t : t = 0, 1, 2, ..., n\}$ of profiles from p to q by setting $r^0 = p$, and for $t > 0$ we let r^t be the profile for which $r^t(i) = r^{t-1}(i)$ for all $i \neq t$, and $r^t(t) = q(t)$. Informally, a *modified* sequence from p to q with J first begins with p and at each stage t we replace one individual relation from p with that same individual's relation at q, *and* we replace $p(i)$ for every $i \in J$ before replacing $p(i)$ for any $i \in N\backslash J$. Formally, $\{r^t\}$ is a *modified sequence* from p to q *with* J *first* if J is a subset of N, and for some one-to-one function $\pi : N \to N$ for which $\pi(\{1, 2, ..., |J|\}) = J$ we have $r^0 = p$, and for $t > 0$ the profile r^t satisfies $r^t(i) = r^{t-1}(i)$ for all $i \neq \pi(t)$, and $r^t(\pi(t)) = q(\pi(t))$. Even when r^0 and r^n both belong to $\mathcal{L}_C$ we can have $r^t \notin \mathcal{L}_C$ for some t, because $\mathcal{L}_C$ is not a product set. We will have to take care in choosing r^0 and r^n in the proof to ensure that each r^t belongs to the domain. (It is easy to see why $\mathcal{L}_C$ is not a product set: For any $x \in X$, any $i \in N$, and *any* relation $\succ$ in $L(X)$, we have $p \in \mathcal{L}_C$ if $p(i) = \succ$ and $p(j)[1] = x$ for all $j \in N\backslash\{i\}$. Therefore, if $\mathcal{L}_C$ were a product set we would have $\mathcal{L}_C = L(X)^N$, which is absurd.)

Although $\mathcal{L}_C$ itself is not a product set, Sen (1966) uncovered a nice product set that is contained in $\mathcal{L}_C$: If n is odd and S is a subset of $L(X)$ then every $p \in S^N$ has a strong Condorcet winner if and only if S is value restricted. A special case of value-restrictedness is single-peakedness on the real line, and that also gives rise to a product set domain. However, on that domain there are many non-dictatorial and strategy-proof social choice rules, including g_C. They are characterized by Moulin (1980), whose results are extended in Barberà, Gul, and Stacchetti (1993).

Manipulation. We say that *coalition* $J \subseteq N$ can manipulate the rule $g : \wp \to X$ at $p \in \wp$ via $q \in A(X)^J$ if the profile r for which $r(j) = q(j)$ for all $j \in J$ and $r(i) = p(i)$ for all $i \in N\backslash J$ belongs to the domain $\wp$ of g, and $g(r) \succ_j^p g(p)$ for all $j \in J$. If $J = \{j\}$, a singleton, we say that individual j can manipulate g at p via $q(j)$.

Strategy-proofness. We say that $g : \wp \to X$ is strategy-proof if no *individual* $i \in N$ can manipulate g at any profile in $\wp$.

We have already demonstrated that $g_C : \wp_C \to X$ is not only strategy-proof, it cannot be manipulated by any coalition. However, for any rule $g : \mathcal{L} \to X$ such that $g(p) = g_C(p)$ for all $p \in \mathcal{L}_C$, we *can* find $p \in \mathcal{L}_C$ and $r \in \mathcal{L}$, and $i \in N$, such that $r(j) = p(j)$ for all $j \neq i$ and $g(r) \succ_i^p g(p)$, as we show in Campbell and Kelly (2002a). When we are referring to the strategy-proofness of a rule with domain $\wp_C$ (resp. $\mathcal{L}_C$), we mean that the profiles of true *and* revealed preferences are both in $\wp_C$ (resp. $\mathcal{L}_C$). We will show that g_C is the only strategy-proof rule on $\wp_C$. But first we will prove that majority rule is the only non-dictatorial and strategy-proof rule on $\mathcal{L}_C$. In both cases we assume that there is an odd number $n > 1$ of individuals. The importance of this is illustrated by Example 1, which shows that for two-person societies there are many non-dictatorial and strategy-proof rules on $\mathcal{L}_C$.

Example 1. $N = \{1, 2\}$ and X is finite. Profile p belongs to $\mathcal{L}_C$ if and only if it belongs to $\mathcal{L}$ and $p(1)$ and $p(2)$ have a common maximal element. In that case *any* $g : \mathcal{L}_C \to X$ for which $g(p)$ is a function of $p(1)[1] = p(2)[1]$ is strategy proof. (That is, we don't necessarily have $g(p) = p(1)[1]$, but $g(r) = g(p)$ will hold if $r(1)$ and $p(1)$ have the same maximal alternative.) To prove this claim, suppose that p and r are any two profiles in $L(X)^{\{1,2\}}$. If $p \in \wp_C$ and $p(i) = r(i)$ for some i, then $r \in \mathcal{L}_C$ if and only if the four relations $r(1), r(2), p(1)$, and $p(2)$ have the same maximal alternative. Hence, an individual cannot manipulate because she cannot report a preference relation with a different maximal element than the other person's reported preference. For that reason, there are many other types of rules on $\mathcal{L}_C$ that are strategy-proof when $n = 2$. For instance, $g(p) = p(1)[2]$, the rule that always selects person 1's second ranked alternative.

The rule that ignores the preferences of the individuals in nonempty coalition J and selects the strong Condorcet winner for the set $N \backslash J$ is strategy-proof. It does not satisfy the hypothesis of our main theorem because it is not defined on all of $\mathcal{L}_C$, as we now demonstrate.

Example 2 (Partial majority rule). Assume that $n \geq 5$ and n is odd. Let $\mathcal{L}_C^3$ be the set of profiles in $\mathcal{L}$ such that there is a strong Condorcet winner for the society $\{1, 2, 3\}$. In other words, for each profile p in $\mathcal{L}_C^3$ there is an alternative x in X such that for each $y \in X/\{x\}$ there are at least two members of $\{1, 2, 3\}$ who strictly prefer x to y. Set $g_C^3(p) = x$ in that case. The rule g_C^3 is strategy proof, but it is not defined on all of $\mathcal{L}_C$. For instance, any profile $p \in \mathcal{L}_C$ for which $p(i)[1] = x$ for $i > 2$ is not in $\mathcal{L}_C^3$ if all the members of X are part of a majority rule cycle for the society $\{1, 2, 3\}$, but it does belong to $\mathcal{L}_C$. Theorem 1 implies that we cannot extend g_C^3 to $\mathcal{L}_C \cup \mathcal{L}_C^3$ without creating an opportunity for manipulation by some individual at some profile.

3 Two majority rule theorems

Recall that we are assuming away individual indifference between distinct alternatives and are confining attention to the case of an odd number of individuals.

Theorem 1. *Assume that $n > 1$ is odd and X has at least three members. If X is finite and $g : \mathcal{L}_C \to X$ is a strategy-proof and non-dictatorial rule with range X, then g is majority rule. If X is infinite and $g : \mathcal{L}_C \to X$ is a strategy-proof rule with range X, then g is majority rule.*

The proof of our theorem involves three steps. The first is a standard one: If everyone reports a binary relation with x on top then x must be selected. The proof is not standard because we have to make sure that each profile that we use belongs to the domain $\mathcal{L}_C$. The middle (and longest) step shows that if g is non-dictatorial then x will be selected when over half of the individuals have x as the maximal element. Once we have established that, it will be easy to prove the final step, showing that g coincides with g_C on $\mathcal{L}_C$ if X is infinite, or if X is finite and g is not dictatorial. Note that if there is a majority coalition with a common maximal alternative at profile p, then p obviously belongs to $\mathcal{L}_C$. In other cases we must explicitly demonstrate that we have not left the domain $\mathcal{L}_C$.

Proof of Theorem 1. Let g be a strategy-proof rule with domain $\mathcal{L}_C$ and range X.

Step 1. Let $p \in \mathcal{L}_C$ be any profile that has x as the maximal element of $p(i)$ for all $i \in N$. We want to show that $g(p) = x$. We know that $g(r) = x$ for some $r \in \mathcal{L}_C$. Suppose that $g_C(r) = x$. Then each member of the standard sequence $\{p^t\}$ from r to p will belong to $\mathcal{L}_C$, for if x is a strong Condorcet winner for p^t then when we replace $p^t(t+1)$ with $p(t+1)$, for which x is a maximal alternative, alternative x must be a strong Condorcet winner for p^{t+1}. Moreover, if $g(p^t) = x$ but $g(p^{t+1}) \neq x$ then individual $t+1$ can manipulate g at p^{t+1} via $p^t(t+1)$. Therefore, induction and strategy-proofness give us $g(p) = g(p^n) = x$.

Suppose, however, that $g_C(r) = y \neq x$. We cannot assume that each member of $\{p^t\}$ belongs to $\mathcal{L}_C$, and we will have to employ another profile $q \in \mathcal{L}_C$ such that $q(i) = (x, y, ...)$ for all $i \in N$. We will first show that $g(q) = x$. Set $J = \{i \in N : x \succ_i^r y\}$. Because x and y are distinct and p belongs to $A(X)^N$ we have $N \backslash J = \{j \in N : y \succ_j^r x\}$. Let $\{r^t\}$ denote a modified sequence from r to q with J first. $N \backslash J$ contains a majority of the members of N because $y = g_C(r)$. This implies that $r^t \in \mathcal{L}_C$ for $0 \leq t < \frac{n+1}{2}$ because y still defeats every other member of X by a majority at r^t. And $r^t \in \mathcal{L}_C$ for $t \geq \frac{n+1}{2}$ because x is the maximal element of $r^t(i)$ for a majority of individuals $i \in N$ when $t \geq \frac{n+1}{2}$. We know that $g(r^0) = x$. If $g(r^t) = x$ but $g(r^{t+1}) \neq x$ then person $t+1$ can manipulate g at r^{t+1} via $r^t(t+1)$, contrary to our supposition. Therefore, we have $g(r^t) = x$ for $t = 0, 1, 2, ..., n$, so $g(q) = g(r^n) = x$. Recall that p is an arbitrarily chosen profile in $\mathcal{L}_C$ for which x is the maximal element for all $i \in N$. Because x is a maximal element of $q(i)$ for each $i \in N$ and $g(q) = x$, we can use the standard sequence from q to p to establish that $g(p) = x$.

Step 2. Now we show that if g is non-dictatorial, we only need x to be a maximal alternative for more than $\frac{n}{2}$ individuals to infer that $g(p) = x$. Let $K = \{i \in N : x$ is maximal for $p(i)\}$. We already know that $g(p) = x$ if $|K| = n$. Now, suppose that $g(p) = x$ whenever $|K| = k$, for $\frac{n+3}{2} \leq k \leq n$. We prove that $g(p) = x$ if $|K| = k - 1$, unless g is dictatorial.

Suppose that alternative x is the maximal element for exactly $k-1$ individuals at p, namely the individuals in J. Suppose that $g(p) = y \neq x$. We will show that this implies that for every z in X there is a profile at which z is selected even though $k-1$ individuals have x as the maximal alternative at that profile. This will allow us to prove that g is dictatorial, and hence we can conclude that $g(p) = x$ if g is not dictatorial. Assuming that $g(p) = y \neq x$, if $x \succ_i^p y$ for some $i \in N \backslash J$, then the induction hypothesis implies that this individual i can manipulate g at p via an ordering that has x as the maximal alternative. Then we must have $y \succ_i^p x$ for all $i \in N \backslash J$.

Let q be any profile in $\mathcal{L}_C$ for which $q(i) = p(i)$ for $i \in J$ and $q(i) = (y, x, ...)$ for $i \in N \backslash J$. Let $\{q^t\}$ be the standard sequence from p to q. We have $g(q^0) = y$. Moreover, $g(q^t) = y$ and strategy-proofness imply $g(q^{t+1}) = y$, because $q^{t+1} \neq q^t$ implies $t+1 \in N \backslash J$, and hence y is a maximal element of $q^{t+1}(t+1)$. Hence, $g(q) = g(q^n) = y$ by induction.

Now, let r be any profile such that $r(i) = (x, ..., y)$ for all $i \in J$ and $r(i) = q(i) = (y, x, ...)$ for $i \in N \backslash J$. Let $\{r^t\}$ be the standard sequence from q to r. Then $g(r^0) = y$. For any t, we have $g(r^t) \in \{x, y\}$ by the induction hypothesis, because any $i \in N \backslash J$ can precipitate the selection of x by reporting any ordering with x as the maximal element. Now, suppose $g(r^t) = y$. If $g(r^{t+1}) = x$ then $t+1 \in J$ and individual $t+1$ can manipulate g at r^t via $r^{t+1}(t+1)$. Hence, $g(r^{t+1}) = y$, so we have $g(r) = g(r^n) = y$.

Let $\wp^*$ denote the set of profiles s in $\mathcal{L}_C$ such that $s(i) = r(i)$ for all $i \in J$ and $s(i) \in L(X)$ for all $i \in N \backslash J$. Define g^* by setting $g^*(s) = g(s)$ for all s in $\wp^*$. We will show that the range of g^* is X. We know that $g(r) = y = g^*(r)$. Also, if $h \in N \backslash J$, if $r'(h) = (x, ...)$, and if $r'(i) = r(i)$ for all $i \neq h$ we have $g(r') = x = g^*(r')$ by the induction hypothesis. Hence, x and y belong to the range of g^*. Now, choose $z \in X \backslash \{x, y\}$. Suppose $s \in \wp^*$ has $s(i) = r(i)$ for $i \in J$ and $s(i) = (z, y, ...)$ for all $i \in N \backslash J$. Let $\{s^t\}$ be the standard sequence from r to s. We have $g(s^0) = y$. Suppose that $g(s^t) = y$ and $g(s^{t+1}) \neq z$. Then $g(s^{t+1}) = y$, otherwise individual $t+1$ could manipulate g at s^{t+1} via $s^t(t+1)$. Therefore, either $g(s^t) = y$ for all t, or else $g(s^t) = z$ for some t. In the latter case we know that z belongs to the range of g^*. Suppose, however, that $g(s^t) = y$ for all t. In particular, $g(s) = y$. Recall that for $i \in J$ we have $s(i) = r(i) = (x, ..., y)$. We next show that $g(s) = y$ leads to a contradiction in that case.

Let u be any profile in $\mathcal{L}_C$ for which $u(i) = (z, ...)$ for all $i \in J$ and $u(i) = s(i) = (z, y, ...)$ for all $i \in N \backslash J$. We have $g(u) = z$ by Step 1. Let $\{u^t\}$ be the standard sequence from s to u. For each t, either a majority of individuals have x as the maximal element (as at s), or else a majority of the individuals have z as the maximal element (as at u). We have $g(u^0) = y$. If $g(u^t) = y$ and $g(u^{t+1}) \neq y$ then $t+1 \in J$ and individual $t+1$ can manipulate g at u^t via $u^{t+1}(t+1)$. Hence, $g(u^t) = y$ for all t, contradicting the fact that $g(u) = z$. Therefore, we have to drop the supposition that $g(s^t) = y$ for all t, and hence we must have $g(s^t) = z$ for some t. We have established that arbitrary $z \in X$ belongs to the range of g^*.

We can use g^* to induce a social choice rule for society $N \backslash J$ and domain $L(X)^{N \backslash J}$ in the natural way, and we also use g^* to denote that induced rule. The main result of Aswal, Chatterji, and Sen (1999) establishes the Gibbard-

Satterthwaite Theorem for the domain $L(X)^H$ and arbitrary finite H. Therefore, some individual $h \in N\backslash J$ is a dictator for g^*. We will show that h is a dictator for g itself. This will establish that either g is dictatorial, or for every $k > \frac{n}{2}$ and arbitrary $p \in \mathcal{L}_C$, if k individuals have a common maximal alternative then that will be the alternative selected by g at profile p.

Choose any $z \in X\backslash\{x, y\}$ and $v \in \mathcal{L}_C$ arbitrarily, except that $v(h)[1] = z, v(h)[2] = x$, and for any $i \in J$ we have $v(i)[1] = x$ and $v(i)[\infty] = z$. Let $v' \in \mathcal{L}_C$ denote the profile for which $v'(i) = r(i)$ for all $i \in J$ and $v'(i) = v(i)$ for all $i \in N\backslash J$. We have $g(v') = z$ because $v' \in \wp^*$ and h is a dictator for g^*. If $\{v^t\}$ is the standard sequence from v' to v we have $g(v^t) \in \{x, z\}$ for all t because individual h can precipitate the selection of x by reporting an ordering with x as the maximal element – that would result in k individuals declaring x to be their maximal alternative. We have $g(v^0) = z$, and if $g(v^t) = z$ then $g(v^{t+1}) = z$ because $g(v^{t+1}) = x$ otherwise, in which case $t+1 \in J$ and individual $t+1$ could manipulate g at v^t via $v^{t+1}(t+1)$. Therefore, $g(v^t) = z$ for all t. In particular, $g(v) = g(v^n) = z$. Because $z = v(i)[\infty]$ for all i in J and $v(i)$ was arbitrarily chosen for all $i \neq h$ in $N\backslash J$ (we could have $v(i)[\infty] = z$), a standard sequence argument will establish $g(v'') = z$ for all $v'' \in \mathcal{L}_C$ such that $v''(h)[1] = z$. For arbitrary $z \in X\backslash\{x, y\}$, we can use the argument of this paragraph, but with the roles of x and z reversed, to show that $g(v'') = x$ for any $v'' \in \mathcal{L}_C$ such that $v''(h)[1] = x$. Similarly, we can establish $g(v'') = y$ for any $v'' \in \mathcal{L}_C$ such that $v''(h)[1] = y$. Therefore, individual h is a dictator for g.

Step 3. We have established by induction that if g is not dictatorial then for arbitrary x in X we have $g(p) = x$ for any profile p in $\mathcal{L}_C$ at which x is the maximal element of the individual ordering of over half of the members of N. Assuming that g is non-dictatorial, it remains to prove that $g(p) = x$ if x defeats every member of $X\backslash\{x\}$ by a clear majority at p. Suppose that this is true whenever l or more individuals have x as the maximal element, but profile p has exactly $\ell - 1$ individuals with $p(i)[1] = x$. Suppose $g(p) = y \neq x$. Then $\ell < \frac{n+1}{2}$, and there exists $i \in N$ such that $p(i)[1] \neq x$ and $x \succ_i^p y$. Let q be the same as p except that $q(i) = (x, ...) \in L(X)$. We have $q \in \mathcal{L}_C$. And $g(q) = x$ by the induction hypothesis, in which case individual i can manipulate g at p. Hence, we must have $g(p) = x$ after all. Then g coincides with g_C on $\mathcal{L}_C$ by induction on ℓ.

All of the above is true whether X is finite or not. But if X is infinite then we can prove that g cannot be dictatorial. Suppose to the contrary that it is dictatorial and that individual h is the dictator. Let $v \in \mathcal{L}_C$ be such that $v(h)$ has no maximal alternative. Then there is some $y \in X$ such $y \succ_h^v g(v)$. Because person h can precipitate the selection of y by reporting a member of $L(X)$ with y as the maximal element and $g_C(v)$ second (the resulting profile will belong to $\mathcal{L}_C$), we have contradicted the supposition that g is strategy-proof. □

The proof of Theorem 1 will go through with domains that are much smaller than $\mathcal{L}_C$, but we have chosen to have an uncomplicated domain so the proof would be fairly transparent. However, we do want to point out that in the case of infinite X, our proof that $g : \mathcal{L}_C \to X$ is majority rule if it is not-dictatorial will go through even if we replace $\mathcal{L}_C$ with $\mathcal{L}_C^m$, for arbitrary finite m, where $\mathcal{L}_C^m$ is the set of profiles p in $\mathcal{L}_C$ at which each individual i has a first-ranked alternative $p(i)[1]$, a second-ranked alternative $p(i)[2]$, a third-ranked alternative $p(i)[3]$, and so on up to m. (One can easily verify this claim for $\mathcal{L}_C^l$.) Restricting the domain to $\mathcal{L}_C^m$ would give us a theorem for domains on which one can adapt the definitions of standard social choice rules to the case of infinite X. For instance, there are a number of ways to adapt the Borda rule to $\mathcal{L}_C^m$. One way is to apply the standard definition to the set

$$\{p(1)[1], p(1)[2], ..., p(1)[m], p(2)[1], p(2)[2], ..., p(2)[m], ..., \\ p(n)[1], p(n)[2], ..., p(n)[m]\}.$$

Then let g select the member of this set with the highest Borda score, using a specific tie-breaking rule when necessary. The resulting social choice rule will not be strategy-proof of course, but this version of Theorem 1 would establish that there is no way of adapting a non-dictatorial rule to the infinite X case, unless it is majority rule or the rule gives some individual an opportunity to manipulate at some profile.

It may be instructive to compare Theorem 1 to the results of Moulin (1980): Let $X = \{x, y, z\}$ and for arbitrary $w \in X$ let $\mathcal{L}^w$ denote the set of profiles p in $\mathcal{L}$ such that $w \neq p(i)[3]$ for any $i \in N$. Then $\mathcal{L}^w$ is the set of profiles of single-peaked preferences when x, y, and z are placed on the real line with w in between the other two members of X. Moulin showed that the strategy-proof rules on $\mathcal{L}^w$ are generalized median voter rules. (He did not restrict the cardinality of X.) The salient case is majority rule, which can be defined on $\mathcal{L}^w$ as the rule that selects the alternative in $\{p(i)[1], p(2)[1], ..., p(n)[1]\}$ that is in the middle, when the alternatives are positioned on the real line in a way that makes $\mathcal{L}^w$ single-peaked. However, given a *particular* positioning of x, y, and z on the real line with w in the middle, the rule g_L that selects the left-most of the individual maximal alternatives is also strategy-proof, as is the rule that selects the right-most alternative. Hence, majority rule is not the only rule on $\mathcal{L}^w$ that is non-dictatorial and strategy-proof. The set $\mathcal{L}^x$ is contained in our domain $\mathcal{L}_C$, but so are $\mathcal{L}^y$ and $\mathcal{L}^z$. Consider the family $\mathcal{L}^y$, with $x < y < z$ on the real line. Although g_L is strategy-proof on $\mathcal{L}^y$, it is not strategy-proof on $\mathcal{L}_C$, as we now show. Suppose $n = 3$ and that profile p has $p(1) = (y, ...) = p(2)$ and $p(3) = (z, x, y)$. Then $g_L(p) = y$. But if person 3 reports the ordering $q(3) = (x, y, z)$ we will have $g_L(q) = x$ if $q(1) = p(1)$ and $q(2) = p(2)$. Note that $g_L(q) \succ_3^p g_L(p)$ and that p and q belong to $\mathcal{L}_C$ because at both profiles alternative y is a maximal element for two individuals. However, p does not belong to $\mathcal{L}^y$. The fact that our domain is much larger than Moulin's accounts for the difference in the results. Moreover, if we replace $\mathcal{L}_C$ with $\wp_C$, then majority rule is the only strategy-proof rule of any kind, as we now prove.

Theorem 2. *Assume that $n > 1$ is odd and that X has at least three members. If $g : \wp_C \to X$ is a strategy-proof rule with range X, then $g = g_C$.*

Proof. The restriction of g to $\mathcal{L}_C$ is obviously strategy-proof, and its range is X by Step 1 of the proof of Theorem 1 because that argument is valid even if r belongs to $\wp_C$ but not to $\mathcal{L}_C$. Therefore, Theorem 1 tells us that the restriction of g to $\mathcal{L}_C$ is either dictatorial or majority rule. Suppose that the restriction of g to $\mathcal{L}_C$ is dictatorial and that individual h is the dictator. Let $v \in \wp_C$ be any profile such that $v(j) \in L(X)$ for all $j \neq h, v(h) \in A(X)$, and for three distinct alternatives a, b, and c in X we have $a \succ_h^v b \succ_h^v c \succ_h^v a$ with $y \succ_h^v z$ for all $y \in \{a, b, c\}$ and $z \in X \backslash \{a, b, c\}$. Then there is an alternative $y \in X$ such that $y \succ_h^v g(v)$, and so individual h can manipulate g at v by reporting a member of $L(X)$ with y as the maximal element and $g_C(v)$ second. We are forced to abandon the assumption that the restriction of g to $\mathcal{L}_C$ is dictatorial. The proof of Theorem 1 can now be applied, with the understanding that g cannot be dictatorial, because the proof only uses profiles in $\mathcal{L}_C$ which is a subset of $\wp_C$. Hence, the restriction of g to $\mathcal{L}_C$ must be majority rule. We conclude by proving that $g = g_C$.

For arbitrary $p' \in \wp_C$ let $m(p')$ denote the number of individuals i such that $p'(i) \notin L(X)$. We know that $g(p') = g_C(p')$ if $m(p') = 0$. Suppose that $g_C(p') = g(p')$ whenever $m(p') \leq m$, but for $p \in \wp_C \backslash \mathcal{L}_C$ we have $m(p) = m + 1$. Let $g_C(p) = x$, and $g(p) = y$. We will show that $x = y$. If $p(i) \notin L(X)$ and $x \succ_i^p y$ then by the induction hypothesis, individual i can manipulate g at p via a member $\succ$ of $L(X)$ such that $\succ[1] = x$. The resulting profile p'' will obviously belong to $\wp_C$ and we will have $x = g_C(p'') = g(p'')$. This contradicts the assumption that g is strategy-proof. Therefore, $p(i) \notin L(X)$ and $x \neq y$ imply $y \succ_i^p x$. Choose any such i and some $q''(i) \in L(X)$ such that $q''(i) = (y, x, ...)$. Set $q''(j) = p(j)$ for all $j \neq i$. We obviously have $q'' \in \wp_C$ and $x = g_C(p'')$. Then $g(q'') = x$ by the induction hypothesis. Then individual i can manipulate g at q'' via $p(i)$, contradicting the assumption that g is strategy-proof. Therefore, we can rule out the possibility that $x \neq y$. By induction on m, we have $g = g_C$. □

The original draft of this paper only had Theorem 2. We are grateful to one of our referees for asking a question that inspired Theorem 1.

4 The dictatorship counterpart to Theorem 2

For the domain $\mathcal{L}$ and finite X with at least three members, there are exactly n different strategy-proof rules with range X; specifically, dictatorship of each of the n individuals. If the domain is expanded to allow for individual indifference, then there can be more than n dictatorial rules. However, there is only one strategy-proof rule with domain $\wp_j = \{p \in A(X)^N : p(j)$ has a maximal element$\}$, as we now show.

Theorem 3. *If X has at least three members and $g : \wp_j \to X$ is a strategy-proof rule with range X, then g is dictatorship of individual j.*

Proof. Let g' be the restriction of g to $\mathcal{L}$. The rule g' is obviously strategy-proof. A simple standard sequence argument (with the sequence lying in $\wp_j$) will show that the range of g' is X. Then g' is dictatorial by the Gibbard-Satterthwaite Theorem. Suppose that individual $h \neq j$ is the dictator. Let p be a member of $\wp_j$ such that $p(i)$ does not have a maximal element for any $i \neq j$. Then there exists some $x \in X$ such that $x \succ_h^p g(p)$. Let q be any profile in $\wp_j$ such that $q(i) = (g(p), ...) \in L(X)$ for all $i \neq h$, and $q(h) = p(h)$. Let $\{q^t\}$ be a modified sequence from p to q with

$N\backslash\{h\}$ first. We have $g(q^0) = g(p)$, and thus $g(p^t) = g(p)$ for all $t \leq n-1$. But then individual h can manipulate g at p^{n-1} by reporting a member of $L(X)$ with x as the maximal element. (The resulting profile will belong to $\mathcal{L}$.) Therefore, individual j must be the dictator for g'. Finally, choose arbitrary $p \in \wp_j$, with x denoting $p(j)[1]$. Choose any $q \in \mathcal{L}$ such that $p(i)[\infty] = x$ for all $i \neq j$ and $q(j)[1] = x$. We have $g(q) = x$, and thus a standard sequence argument will reveal that $g(p) = x$. Therefore, individual j is a dictator for g. □

5 Conclusion

Our results are based on domains that are not product sets, and that means that an individual cannot know if his reported preference ordering is admissible without knowing what others will report. That limits the applicability of our results. However, there is a sense in which results based on a domain of single-peaked preferences have the same drawback: Although single-peaked domains can be defined as product sets, single-peakedness is characterized by means of a particular linear ordering, and an individual would have to know the linear ordering according to which the reported preferences of other voters are admissible, before being convinced that his own reported preference is admissible. Moreover, it is possible to apply the definition of single-peakedness and value restriction to individual profiles, and in either case the condition implies the existence of a majority winner. The collection of all single-peaked (resp., value restricted) *profiles* is not a product set.

One implication of Theorem 1 is that the results of Moulin (1980) and of Barberà, Gul, and Stacchetti (1993) are not robust to significant expansions in their domains, even when the existence of a strong Condorcet winner is preserved. These two papers characterize the family of strategy-proof and non-dictatorial rules on the given domain – there are many members of the family, and each is a generalized median voter scheme.

We wonder if the following is true: For arbitrary $\wp \subseteq A(X)^N$, if $g : \wp \to X$ is the only non-dictatorial and strategy-proof rule with domain $\wp$ and range X then g is partial majority rule. (Partial majority rule ignores the preferences of some (possibly empty) subset J of individuals, and applies majority rule to the society $N\backslash J$. Dictatorship is a special case.) There would be an immediate corollary: If g has range X and is the only strategy-proof rule with domain $\wp$ that is not independent of the preferences of any individual then g always selects the strong Condorcet winner. For both the conjecture and the corollary, one would have to include the assumption that $\wp$ is a subset of $A(X)^N$, otherwise serial dictatorship would provide a counterexample. (Serial dictatorship selects the maximal alternative of person 1, say, but if the maximal element is not unique then the preference relation of person 2 is used to eliminate one or more of the alternatives, and if the result is still not a singleton set, use the preferences of person 3, and so on.)

We do not know if Theorem 1 goes through if there is an even number $n \geq 4$ voters, or if individuals can be indifferent between distinct alternatives. In neither case do we have an example of a rule that is strategy-proof, other than g_C, nor do we have a proof that g_C is the only non-dictatorial and strategy-proof rule. However, for even n we have been able to prove that if $g : \mathcal{L}_C \to X$ is non-dictatorial and cannot be manipulated by individuals *or* two-person coalitions then g is majority rule. For n odd, the proof of Theorem 1 does not depend on invulnerability to manipulation by coalitions of two or more persons. However, the hypothesis of our theorem establishes that no coalition of any size can manipulate g because it implies that $g = g_C$, and no coalition can manipulate g_C within $\wp_C$.

Individual preference relations that are not transitive played a role in the proof only for finite X, and then only when we eliminated dictatorial rules.

Obviously, the assumption that the range X_g of g is X is not essential to either theorem. As long as X_g has three or more alternatives, the proofs go through with X_g substituting for X. Campbell and Kelly (2002b) have a general theorem on the addition of variables that included in the set of alternatives that are ordered by individuals but which are never selected by the social choice rule.

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